Math 7800: K-stability

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Preface

These are lecture notes for Math 7800 taught at the University of Utah in the Fall of 2022. The course will be on K-stability with a focus on recent progress on understanding the K-stability of Fano varieties using algebraic tools.

This document will be updated weekly throughout the semester. Any comments, corrections, and questions on these notes are welcome.

The prerequisite for this course will be a solid foundation in algebraic geometry. For example, a year long course in algebraic geometry or Chapters I-V of Hartshorne will suffice. Knowledge of some higher dimensional geometry, such as material on positivity of line bundles [**Laz04**, Sections 1.1-1.5] and singularities of the Minimal Model Program [**KM98**, Section 2.3], will be helpful, but not strictly necessary. We will review the relevant definitions and concepts in class.

CHAPTER 1

Introduction

In this chapter, we give a brief survey on: “What is K-stability?” This is meant to provide motivation and context for definitions and topics that will appear in future chapters.

K-stability is a notion that was introduced by Tian to detect the existence of certain canonical metrics on algebraic varieties [Tia97]. The definition was reformulated in a purely algebraic way and generalized by Donaldson [Don02]. While the latter definition is algebraic, it originally looked quite foreign to algebraic geometers. For example, the definition is quite involved to state and difficult to verify in even the simplest of examples without additional machinery. Hence, algebraic geometers originally did not expect the notion to be important outside of complex differential geometry.

Recently, there has been great interest and progress by the algebraic geometry community in understanding K-stability algebraically. This has relied on connections between K-stability and topics including birational geometry, non-Archimedean geometry, and singularity theory. This progress has also resulted in applications to algebraic geometry: the construction of moduli spaces of Fano varieties. The goal of this course is to understand some of this algebraic progress and applications.

Conventions: In this chapter, all varieties are defined over the field $\mathbb{C}$, since we will be discussing connections between complex differential geometry and algebraic geometry.

1. Kähler–Einstein metrics

The motivation for K-stability arises from the search for canonical Kähler metrics on algebraic varieties. Below, we briefly discuss the key points in the theory.

1.1. Kähler metrics. On a complex manifold X, a natural class of metrics to consider are Hermitian metrics. Such a metric ω is the data of a (1,1) form, which can be described in local holomorphic coordinates $z_1, \dots, z_n$ by

$$\omega =_{\text{locally}} \frac{i}{2\pi} \sum h_{ij} dz_i \wedge d\overline{z}_j,$$

where h_{ij} is a Hermitian matrix of $\mathbb{C}$-valued C^∞ functions on X. A Hermitian metric is *Kähler* if $d\,\omega = 0$.

The simplest example of a Kähler metric is the *Fubini–Study* ω_{FS} metric on $\mathbb{CP}^n$. In the coordinates $z_1, \dots, z_n$ on the affine chart $\{[1 : z_1 : \dots : z_n] \mid z_i \in \mathbb{C}\} \subset \mathbb{P}^n$,

$$\omega_{\mathrm{FS}} =_{\text{locally}} \frac{i}{2}\partial\overline{\partial}\log\left(\sum |z_i|^2 + 1\right).$$

An elegant feature of this metric is that it is invariant with respect to the action by $\mathrm{SU}(n+1)$ on $\mathbb{P}^n$. Though, since it is not invariant with respect to $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}(n+1)$, it is not entirely canonical without the choice of coordinates!

On any smooth projective complex variety X, we can always construct a Kähler metric on X as follows. For any embedding $X \subset \mathbb{P}^n$, the pull-back of the Fubini–Study metric $\omega := \omega_{FS}|_X$ is a Kähler metric on X. Since X can be embedded in bigger and bigger projective spaces, we should expect the space of Kähler metrics on X to be infinite dimensional.

1.2. Kähler-Einstein metrics. In order to find canonical metrics on a smooth complex algebraic variety, one must consider a more restrictive class of metric. A Kähler–Einstein metric ω is a Kähler metric such that

$$\mathrm{Ric}(\omega) = \lambda\omega$$

for some $\lambda \in \{-1, 0, 1\}$. Above, $\mathrm{Ric}(\omega)$ denotes the Ricci curvature of ω, which is the (1,1) form locally given by $\mathrm{Ric}(\omega) {=_{\text{locally}}} i\sum_{i,j} R_{ij} dz_i \wedge d\overline{z}_j$, where $R_{ij} = -\frac{\partial^2}{\partial z_i \partial \overline{z}_j}\log\det h$. The simplest examples of a Kähler-Einstein metric is the Fubini–Study metric on $\mathbb{CP}^n$.

Not all complex varieties can admit a Kähler-Einstein metrics. Indeed, the Kähler-Einstein equality implies either

(1) X is canonically polarized (i.e. K_X is ample);
(2) X is K-trivial (i.e. $K_X \sim_{\mathbb{Q}} 0$)
(3) X is Fano (i.e. $-K_X$ is ample).

The geometry of X is drastically different in each of the above three cases.

An amazing feature of Kähler–Einstein metrics is that when they exist they are unique up to action by $\mathrm{Aut}(X)$. Hence, they may be viewed as "canonical" metrics. This leads to the question of when such a Kähler-Einstein metric exists.

In landmark papers of Yau and Yau and Aubin it was shown that Calabi-Yau varieties and canonically polarized varieties always admit Kähler-Einstein metrics [**Aub78**, **Yau78**]. The case of Fano varieties turns out to be surprisingly more subtle and interesting.

Not all Fano varieties admit Kähler–Einstein metrics. For example, it was shown by Matsushima that Kähler-Einstein Fano varieties must have reductive automorphism group [**Mat57**]. Hence, $B_p\mathbb{P}^2$ cannot admit a Kähler-Einstein metric. This leads to the question of which Fano varieties admit Kähler-Einstein metrics.

1.3. Examples. It has long been a difficult problem to determine if a given Fano variety variety admits a Kähler–Einstein metric.[1] Here we list some basic examples where the answer is known.

(1) Projective space: The Fubini–Study metric ω_{FS} is a KE metric on $\mathbb{P}^n$.
(2) Curves: A smooth projective curve C is Fano if and only if $C \simeq \mathbb{P}^1$. The latter always admits a Kähler-Einstein metric by the previous example.
(3) Del Pezzo surfaces: A *del Pezzo surface* is a smooth projective surface X with $-K_X$ ample (in other words, a Fano variety of dimension 2). By classification results, such varieties are of the form

$$\mathbb{P}^2, \quad \mathbb{P}^1 \times \mathbb{P}^1, \quad B_{p_1,\ldots,p_r}\mathbb{P}^2$$

where the last variety is the blowup of $\mathbb{P}^2$ at $r \leq 8$ sufficiently general points. All such varieties admit KE metrics, except $B_{p_1}\mathbb{P}^2$ and $B_{p_1,p_2}\mathbb{P}^2$.
(4) Fano threefolds: By the work of Iskovskih, Mori, and Mukai, Fano 3-folds admit a classification into 105 types. There has been recent work in analyzing which 3-folds admit Kähler-Einstien metrics [**ACC+21**].
(5) Hypersurfaces: If $X \subset \mathbb{P}^n$ is a smooth projective hypsurface of degree d, then adjunction implies

$$\omega_X = \mathcal{O}_X(d - n - 1).$$

Hence, X is Fano if $d < n + 1$. By work of Tian and Donaldson, a general Fano hypsurface admits a KE metric . It is a folklore conjecture that *all* Fano hypersurfaces admit KE metrics; see [**Fuj19**, **AZ21**] for progress on this problem.

1.4. Geometry of Fano varieties. Since Fano varieties play an important role in K-stability and these notes, we give a brief digression on their properties.

A key feature of Fano varieties is there particularly "well behaved" geometry. In particular, they satisfy the following properties:

(1) Fano varieties are *rationally connected*. The latter means: if x and y are two points on a Fano variety X, then there exists a morphism $f : \mathbb{P}^1 \to X$ such that $f(0) = x$ and $f(1) = y$.
(2) Fano varieties are simply connected.
(3) On a Fano variety X, $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.[2]
(4) On a Fano variety X, $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$. Indeed, consider the exponential sequence

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}_X^* \to 0.$$

[1]Recently using recent progress in algebraic K-stability theory, the problem is often solvable in concrete examples in dimension ≤ 3.

[2]The result follows from Kodaira vanishing, which states that if X is a smooth complex projective variety and L an ample line bundle on X, then $H^i(X, \omega_X \otimes L) = 0$ for all $i > 0$. Applying the theorem to a Fano variety X with $L = \omega_X^*$ gives the desired result.

Taking cohomology gives an exact sequence

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$$

is exact. Using that $H^1(X, \mathcal{O}_X^*) \simeq \mathrm{Pic}(X)$ and (3), we conclude $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$.

(5) Fano varieties of fixed dimension n form a bounded family. This means that for each integer $n > 0$, there exists a smooth projective morphism of varieties $Y \to S$ such that: if X is a Fano variety of dimension n, then $X \simeq Y_s$ for some $s \in S$.

Note that the above properties do not hold for canonically polarized.

While the geometry of Fano varieties is well-behaved, this does not mean that they are easy to understand. For example, it is an open problem to determine which Fano hypersurfaces rational. Even the seemingly simple case of a general cubic fourfold is wide open. Hence, the geometry of Fano varieties is very rich.

2. K-stability

2.1. Yau-Tian-Donaldson Conjecture. In the early 1990s, Yau conjectured that there should exist an algebraic stability notion that detects when a Fano variety admits a Kähler-Einstein metric. The latter was inspired by the Kobayashi–Hitchin correspondence, which relates the existence of a certain metric on a vector bundle over a smooth projective variety to the slope stability of the vector bundle.

In order to make Yau's conjecture precise, Tian defined K-stability [**Tia97**]. In [**Don02**], Donaldson reformulated K-stability using a purely algebraic definition and generalized the notion. Hence, the following statement, which is now a theorem, is often referred to the Yau–Tian–Donaldson Conjecture:

Theorem 2.1 (Chen–Donaldson–Sun and Tian). *A smooth Fano variety X admits a Kähler–Einstein metric if and only if it is K-polystable.*

The forward implication of the theorem, often considered the easier implication, was proven by Berman [**Ber16**]. The reverse implication was shown independently by Chen, Donaldson, and Sun in [**CDS15**] and Tian [**Tia15**]. More recently, there have been additional proofs of the theorem, as well as an extension of the above theorem to singular Fano varieties [**LTW21**, **Li22**, **LXZ22**].

2.2. Definition. K-stability is defined in terms of the positivity of a numerical invariant on the set of equivariant degenerations of the variety. We sketch the definition below and will include more details in future chapters.

Throughout, let X be a projective variety and L an ample line bundle on X. An important special case is when X is a Fano variety and $L = \omega_X^*$.

2.2.1. *Test configurations.* A test configuration of (X, L) is a type of $\mathbb{C}^*$-equivariant degeneration of the pair. To construct a test configuration, fix

(1) an embedding $X \xhookrightarrow{|L^k|} \mathbb{P}^N$ for some $k > 0$ and

(2) a group homomorphism $\rho : \mathbb{C}^* \to \mathrm{PGL}(N+1, \mathbb{C})$.

Note that ρ induces an action of $\mathbb{C}^* \curvearrowright \mathbb{P}^N$ and, hence, induces an embedding

$$j : X \times \mathbb{G}_m \hookrightarrow \mathbb{P}^N \times \mathbb{G}_m \quad \text{defined by} \quad (x,t) \mapsto (\rho(t)x, t).$$

Consider the closure

$$\mathcal{X} := \overline{j(X \times \mathbb{G}_m)} \subset \mathbb{P}^N \times \mathbb{A}^1 \quad \text{and} \quad \mathcal{L} := \mathcal{O}_{\mathcal{X}}(1).$$

By [**Har77**, Proposition 9.8], the morphism $\mathcal{X} \to \mathbb{A}^1$ is flat. Hence, $\mathcal{X}_0$ is sometimes referred to as the flat limit of $\rho(t) \cdot X \subset \mathbb{P}^N$ as $t \to 0$. The data of $(\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ is a test configuration.[3]

Example 2.2. We give a simple example of a test configuration of $(X, \mathcal{O}_X(1))$. Consider the embedding $\mathbb{P}^1 \times \mathbb{P}^1 \simeq \{w^2 + x^2 + y^2 - z^2 = 0\} \subset \mathbb{P}^3_{w,x,y,z}$ and the action $\rho : \mathbb{C}^* \to PGL(4)$, where $\rho(t) \cdot [w : x : y : z] = [w : x : y : t^{-1}z]$. Since

$$\rho(t) \cdot \{w^2 + x^2 + y^2 + z^2 = 0\} = \{w^2 + x^2 + y^2 - t^2 z^2 = 0\},$$

it follows that

$$\mathcal{X} := \{w^2 + x^2 + y^2 - t^2 z^2 = 0\} \subset \mathbb{P}^2_{w,x,y,z} \times \mathbb{A}^1_t$$

Observe that $\mathcal{X}_t \simeq \mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq 0$, while $\mathcal{X}_0 = \{x^2 + y^2 - z^2\} \subset \mathbb{P}^3$, which is the cone over a conic.

2.2.2. *Futaki invariant.* Associated to a test configuration $(\mathcal{X}, \mathcal{L})$ is the Futaki invariant, denoted $\mathrm{Fut}(\mathcal{X}, \mathcal{L})$. It is defined in terms of the action

$$\mathbb{C}^* \curvearrowright H^0(\mathcal{X}_0, \mathcal{L}_0^m)$$

as $m \to \infty$. This algebraic definition was introduced in [**Don02**].

2.2.3. *Definition of K-stability.* Following [**Don02**], we say (X, L) is:

(1) *K-semistable* if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations $(\mathcal{X}, \mathcal{L})$ of (X, L);
(2) *K-polystable* if it is K-semistable and $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = 0$ only when $(\mathcal{X}, \mathcal{L}) \simeq (X, L) \times \mathbb{A}^1$.

The above definition is modeled on GIT, which is defined in terms of the non-negativity of a numerical invariant along $\mathbb{G}_m$-equivariant degenerations. A key difference between K-stability and GIT is that in K-stability the set of degenerations considered is not finite dimensional. Hence, K-stability is sometimes imprecisely referred to as an infinite dimensional GIT problem.

[3] The previous construction is not the standard definition of a test configuration, which is defined more abstractly. The more abstract definition will be defined in Chapter 2 of these notes.

2.3. Recent progress. In the past decade, there has been significant progress in understanding K-stability, especially in the case Fano varieties. We summarize a number of the important results below:

(1) *Singularities*: While K-stability is a global condition on (X, L), Odaka showed that it imposes conditions on the singularities of X [**Oda13**]. More specifically, if (X, L) is K-semistable and some mild additional assumptions are satisfied, then X has lc singularities, which are a class of singularities appearing in the Minimal Model Program.
(2) *Special test configurations*: Li and Xu showed that to test K-stability of a Fano variety X, it suffices to consider test configurations $\mathcal{X}$ such that $\mathcal{X}_0$ has klt singularities [**LX14**]. As above, klt singularities are a class of mild singularities that appear in the MMP.
(3) *Valuative criterion*: Fujita and Li gave a criterion to test K-stability using valuations, rather than test configurations [**Fuj19**, **Li17b**].
(4) *Normalized volume*: Motivated by a local construction in differential geometry, Li introduced the normalized volume of a klt singularity [**Li17b**]. The K-stability of a Fano variety X can be detected by the normalized volume of its cone $\operatorname{Spec} \oplus_{m\geq 0} H^0(X, -mK_X)$.
(5) *K-moduli of Fano varieties*: Using recent advances in understanding the K-stability of Fano varieties, a large group of mathematicians have constructed compact moduli spaces parametrizing K-polystable Fano varieties with at worst klt singularities.

A number of these advances will be discussed in future chapters.

CHAPTER 2

K-stability of polarized varieties

In this chapter, we first carefully define K-stability in terms of test configurations and Donaldson's version of the Futaki invariant. Next, we prove Odaka and Wang's formula for the Futaki invariant in terms of intersection numbers and apply it to prove results of Odaka on K-stability of polarized varieties.

References: K-stability was first introduced for Fano varieties by Tian [**Tia97**]. In his work, test configurations were always assumed to have normal special fiber and the Futaki invariant was defined analytically.

In [**Don02**], Donaldson defined K-stability for polarized schemes. In his work, test configuration were allowed to have arbitrarily bad special fiber and the Futaki invariant was defined algebraically in terms of weights. In this section, we use Donaldson's definition, but closely following the presentation in [**BHJ17**, Sections 1-3], which is an excellent reference.

Conventions:

(1) Unless stated otherwise, all schemes are defined over an algebraically closed field k of arbitrary characteristic.
(2) A polarized pair (X, L) is the data of a scheme X and an ample line bundle L on X.

1. Test configurations

1.1. $\mathbb{Q}$-line bundles. In the definition of K-stability, it will be convenient to work with $\mathbb{Q}$-line bundles, rather than line bundles. This definition allows us to scale line bundles by rational numbers.

Definition 1.1. A $\mathbb{Q}$-*line bundle* L on a scheme X is a formal symbol

$$L = M^{\otimes 1/m}$$

where M is a line bundle on X and m a positive integer. Two $\mathbb{Q}$-line bundles $L = M^{\otimes 1/m}$ and $L = M'^{\otimes 1/m'}$ are *equivalent* if

$$M^{\otimes d/m} \simeq M^{\otimes d/m'}$$

are isomorhic as line bundles for some $d > 0$ sufficiently divisible. Equivalently, the $\mathbb{Q}$-line bundles agree as elements of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

To see the use of this definition, consider the following example.

EXAMPLE 1.2. Let $A = \bigoplus_{m\in\mathbb{N}} A_m$ be a finitely generated graded k-algebra with $A_0 = k$ and set $X = \mathrm{Proj}(A)$.

(1) In general, $\mathcal{O}_X(1)$ is not necessarily a line bundle. For example, take $A = k[x,y]$, where x and y have weights 2.
(2) For a sufficiently divisible positive integer m, $\mathcal{O}_X(m)$ will be a line bundle and, hence, $\mathcal{O}_X(m)^{\otimes 1/m}$ is a $\mathbb{Q}$-line bundle.

At times this notion can cause confusion. Since most constructions in K-stability are only concerned with high powers of a line bundle, it will not cause technical problems.

1.2. Definition of test configurations. In the definition of K-stability, we consider equivariant degenerations of a polarized pair over the affine line. These degenerations are called test configurations and defined as follows.

DEFINITION 1.3. Let X be a projective scheme and L an ample line bundle on X. A *test configuration* of $(\mathcal{X},\mathcal{L})$ of (X,L) is the data of

(1) a proper morphism of schemes $\mathcal{X} \to \mathbb{A}^1$,
(2) a $\mathbb{G}_m$-action on $\mathcal{X}$ extending the standard $\mathbb{G}_m$-action on $\mathbb{A}^1$,
(3) a $\mathbb{G}_m$-linearized ample $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$,
(4) an isomorphism $(\mathcal{X}_1,\mathcal{L}_1) \simeq (X,L)$.

REMARK 1.4. The isomorphism in (4) and the $\mathbb{G}_m$-action on $\mathcal{X}$ induces a $\mathbb{G}_m$-equivariant isomorphism

$$(\mathcal{X},\mathcal{L})_{\mathbb{A}^1\setminus 0} \simeq (X,L)\times(\mathbb{A}^1\setminus 0),$$

where $\mathbb{G}_m$ acts on the right hand side as the product of the trivial action on (X,L) and the standard action on $\mathbb{A}^1\setminus 0$. The latter isomorphism induces a birational morphism

$$\mathcal{X} \dashrightarrow X\times(\mathbb{A}^1\setminus 0)$$

As we will later see, understanding test configurations will be related to understanding $\mathbb{G}_m$-equivariant birational models of $X\times\mathbb{A}^1$

REMARK 1.5. If $(\mathcal{X},\mathcal{L})$ is a test configuration of (X,L), we can *twist* the test configuration as follows. For each integer d, $t^{-d}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(d\mathcal{X}_0)$ is a $\mathbb{G}_m$-linearized line bundle.[1] Hence, the pair $(\mathcal{X},\mathcal{L}\otimes\mathcal{O}_{\mathcal{X}}(d\mathcal{X}_0))$ is a test configuration of (X,L) and differs from $(\mathcal{X},\mathcal{L})$ by the linearization.

EXAMPLE 1.6. We list two related examples of test configurations.

(1) *The trivial test configuration* is

$$(X_{\mathbb{A}^1}, L_{\mathbb{A}^1}) := (X,L)\times\mathbb{A}^1,$$

where $\mathbb{G}_m$-acts as the product of the trivial action on (X,L) and the standard action on $\mathbb{A}^1$.

[1]Note that $\mathcal{O}_{\mathcal{X}}(c\mathcal{X}_0) \simeq \mathcal{O}_{\mathcal{X}}$ as line bundles, but their natural linearizations differ.

(2) Fix a $\mathbb{G}_m$-action on X and a $\mathbb{G}_m$-linearization of L. The data induces a *product test configuration*
$$(X_{\mathbb{A}^1}, L_{\mathbb{A}^1}) := (X, L) \times \mathbb{A}^1$$
where $\mathbb{G}_m$-acts diagonally on $X_{\mathbb{A}^1}$ as the product of the $\mathbb{G}_m$-action on X and the standard action on $\mathbb{A}^1$. Note that this differs from (1), since the $\mathbb{G}_m$-action is not necessarily trivial.

Example 1.7 (Deformation to the normal cone). The following construction, which plays a key role in Fulton's Intersection Theory [**Ful98**], provides many examples of non-trivial test configurations.

Fix a closed subscheme $Z \subseteq X$ and consider the blowup
$$\mathcal{X} := \mathrm{Bl}_{Z\times 0}(X \times \mathbb{A}^1) \xrightarrow{p} X \times \mathbb{A}^1.$$
The induced morphism $\mathcal{X} \to \mathbb{A}^1$ is proper, since it is a composition of proper morphisms, and flat by [**Ful98**, Appendix B.6.7]. Additionally,
$$\mathcal{X}_0 = E + F,$$
where $E = p^{-1}(Z \times 0)$ is the exceptional divisor and F is the strict transform of $X \times 0$. See [**Ful98**, Section 5.1]

We claim that $\mathcal{X}$ can be endowed with the structure of a test configuration. Since $Z \times 0$ is a $\mathbb{G}_m$-invariant subscheme of $X \times \mathbb{A}^1$, there is an induced $\mathbb{G}_m$-action on $\mathcal{X}$. Since $-E$ is ample over $X \times \mathbb{A}^1$, $\mathcal{L}(-tE)$ is an ample $\mathbb{Q}$-line bundle for $0 < t \ll 1$. Hence, $(\mathcal{X}, \mathcal{L}(-tE))$ a test configuration of (X, L)

Example 1.8 (Degenerations in projective space). Fix an integer r such that rL is very ample and consider the embedding $X \hookrightarrow \mathbb{P}(V^*)$ where $V := H^0(X, rL)$. A group homorphism $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$ induces a $\mathbb{G}_m$-action
$$\mathbb{P}(V^*) \times \mathbb{A}^1.$$
Let $\mathcal{X}$ denote the closure of the image of
$$X \times \mathbb{G}_m \hookrightarrow \mathbb{P}(V^*) \times \mathbb{G}_m$$
under the map (x, t) maps to $(\rho(t)x, t)$. By [**Har77**, Proposition 9.8], $\mathcal{X} \to \mathbb{A}^1$ is flat and, hence, $\mathcal{X}_0$ is often referred to the flat limit of $\rho(t) \cdot X$ as $t \to 0$. Additionally, the $\mathbb{G}_m$-action on $\mathbb{P}(V^\times) \times \mathbb{A}^1$ restricts to a $\mathbb{G}_m$-action on $\mathcal{X}$. Hence, $(\mathcal{X}, \frac{1}{k}\mathcal{O}_{\mathcal{X}}(1))$ is a test configuration of (X, L). As discussed in [**BHJ17**, Section 2.3], all test configurations of (X, L) arise from this construction.

1.3. Rees correspondence. The Rees correspondence provides a way to understand $\mathbb{G}_m$-linearized vector bundles on $\mathbb{A}^1$. Before discussing this, we start with a simpler correspondence.

Example 1.9 ($\mathbb{G}_m$-actions on vector spaces). Given a a vector space V, there is a bijective correspondence
$$\mathbb{G}_m\text{-actions on } V \quad \longleftrightarrow \quad \mathbb{Z}\text{-gradings of } V.$$

Indeed, if $\mathbb{G}_m$-acts on V, there is a weight decomposition $V = \oplus_{\lambda\in\mathbb{Z}} V_\lambda$, where

$$V_\lambda := \{v \in V \mid a \cdot v = a^\lambda v \text{ for all } a \in \mathbb{G}_m(k)\}$$

is the λ-weight space. Conversely, given a decomposition $V = \oplus_{\lambda\in\mathbb{Z}} V_\lambda$ as a direct sum of vector spaces, we can define a $\mathbb{G}_m$-action on V by

$$a \cdot v = \sum a^\lambda \cdot v_\lambda,$$

where $v = \sum v_\lambda$ and $v_\lambda \in V_\lambda$.

For the Rees correspondence, we will use the notion of a filtration.

Definition 1.10 (Filtrations). A *$\mathbb{Z}$-filtration* $F^\bullet$ of a finite dimensional vector space V is the data of subspaces $F^\lambda V \subset V$ for each $\lambda \in \mathbb{Z}$ such that

(1) $F^{\lambda+1}V \subset F^\lambda V$ for each $\lambda \in \mathbb{Z}$,
(2) $F^\lambda V = 0$ for $\lambda \gg 0$, and
(3) $F^\lambda V = V$ for $\lambda \ll 0$.

The Rees construction states that there is a correspondence:

$$\mathbb{G}_m\text{-linearized vector bundles on } \mathbb{A}^1 \quad \longleftrightarrow \quad \mathbb{Z}\text{-filtrations of of vector spaces.}$$

(On both sides, we only work with finite dimensional vector spaces and vector bundles.) We first explain the reverse direction of the correspondence, which uses Rees algebras.

Definition 1.11 (Rees algebra). The *Rees algebra* of a $\mathbb{Z}$-filtration $F^\bullet$ of V is

$$\mathrm{Rees}(F^\bullet) := \bigoplus_{\lambda\in\mathbb{Z}} F^\lambda V t^{-\lambda}.$$

Note $\mathrm{Rees}(F^\bullet)$ is a finitely generated torsion free $k[t]$-module, where the module structure is given by $t(vt^{-\lambda}) = vt^{-\lambda+1}$. Additionally, $\mathrm{Rees}(F^\bullet)$ admits a $\mathbb{Z}$-grading that respects the $\mathbb{Z}$-grading on $k[t]$. Hence, $\mathrm{Rees}(F^\bullet)$ corresponds to a $\mathbb{G}_m$-linearized vector bundle $\mathcal{V}$ on $\mathbb{A}^1$.

For the forward direction of the correspondence, fix a $\mathbb{G}_m$-linearized vector bundle $\mathcal{V}$ on $\mathbb{A}^1$. Since the space of global sections admits a $\mathbb{G}_m$-action, there is a decomposition into subspaces

$$H^0(\mathbb{A}^1, \mathcal{V}) = \bigoplus_{\lambda\in\mathbb{Z}} H^0(\mathbb{A}^1, \mathcal{V})_\lambda.$$

Since t has weight -1, multiplication by t induces a map

$$H^0(\mathbb{A}^1, \mathcal{V})_\lambda \xrightarrow{\cdot t} H^0(\mathbb{A}^1, \mathcal{V})_{\lambda-1}.$$

Since $H^0(\mathbb{A}^1, \mathcal{V})$ is a torsion free $k[t]$-module, the map $\cdot t$ is an inclusion. To define a $\mathbb{Z}$-filtration, set $V := \mathcal{V}_1$[2] and

$$F^\lambda V := \operatorname{image}\left(H^0(\mathbb{A}^1, \mathcal{V})_\lambda \to V\right)$$

Proposition 1.12. *With the notation in the above paragraph, the following hold:*

(1) $F^\bullet$ is a $\mathbb{Z}$-filtration of V.
(2) The natural map $H^0(\mathbb{A}^1, \mathcal{V}) \to \operatorname{Rees}(F^\bullet)$ is an isomorphism of $\mathbb{Z}$-graded $k[t]$-modules.

Proof. Note that the diagram

$$\begin{array}{ccc} H^0(\mathbb{A}^1, \mathcal{V})_\lambda & \xrightarrow{\cdot t} & H^0(\mathbb{A}^1, \mathcal{V})_{\lambda-1} \\ \downarrow & & \downarrow \\ V & \xrightarrow{=} & V \end{array}$$

commutes, since the image of t under the morphism $k[t] \to k[t]/(t-1) \cong k$ is 1. Hence, the fact that the top row is an inclusion implies $F^\lambda V \subset F^{\lambda-1}V$. Since $H^0(\mathbb{A}^1, \mathcal{V})$ is a finitely generated $k[t]$-algebra, it follows that $H^0(\mathbb{A}^1, \mathcal{V})_\lambda = 0$ for $\lambda \gg 0$ and, hence, $F^\lambda V = 0$ for $\lambda \gg 0$. Since $\operatorname{im}(H^0(\mathbb{A}^1, \mathcal{V}) \to V) = V$, $F^\lambda V = V$ for $\lambda \ll 0$. This completes the proof of (1).

For the proof of (2), we first claim that the natural map

$$H^0(\mathbb{A}^1, \mathcal{V})_\lambda \to V$$

is injective. If not, then there exists $0 \neq v_\lambda \in H^0(\mathbb{A}^1, \mathcal{V})$ such that $v_\lambda \in (t-1)H^0(\mathbb{A}^1, \mathcal{V})$. This is not possible, since there are no nonzero elements of $(t-1)H^0(\mathbb{A}^1, \mathcal{V})$ of pure degree λ. Hence, the map is injective and the natural map $H^0(\mathbb{A}^1, \mathcal{V})_\lambda \to F^\lambda V$ is an isomorphism. the latter implies (2) holds. $\square$

Proposition 1.13. *If $\mathcal{V}$ is a $\mathbb{G}_m$-linearized vector bundle on $\mathbb{A}^1$ and $F^\bullet$ is the corresponding $\mathbb{Z}$-filtration of $V := \mathcal{V}_1$, then we have $\mathbb{G}_m$-equivariant isomorphisms*

$$\mathcal{V}_{\mathbb{A}^1\setminus 0} \simeq V \times (\mathbb{A}^1 \setminus 0) \quad \text{and} \quad \mathcal{V}_0 \simeq \operatorname{gr}_F^\bullet V = \bigoplus_{\lambda\in\mathbb{Z}} F^\lambda V/F^{\lambda+1}V.$$

Proof. Using the Rees correspondence, this amounts to computing certain tensor products of modules. For the first isomorphism, consider the map of $k[t, t^{-1}]$-modules

$$\phi : \bigoplus_{\lambda\in\mathbb{Z}} F^\lambda V t^{-\lambda} \otimes_{k[t]} k[t, t^{-1}] \to \bigoplus_{\lambda\in\mathbb{Z}} V t^{-\lambda}.$$

The map is clearly injective. To see it surjective, fix an element $st^{-\lambda}$, where $s \in V$. Since $F^\bullet$ is a filtration, there exists $\mu \in \mathbb{Z}$ such that $s \in F^\mu V$. Using that $\phi(s^\mu t^{-\mu} \otimes t^{\mu-\lambda}) = st^{-\lambda}$, we conclude ϕ is surjective.

[2]The notation here is slightly abusive, since $\mathcal{V}_1$ is technically a locally free sheaf on $\operatorname{Spec}(k)$, which we identity with a vector space V. We make similar abusives of notation throughout this section.

To prove the second isomorphism, we must show

$$\mathrm{Rees}(F^\bullet) \otimes_{k[t]} k[t]/(t) \simeq \mathrm{gr}_F(V)$$

are isomorphic as $\mathbb{Z}$-graded k-vector spaces. This is clear, since $t \cdot \mathrm{Rees}(F^\bullet) = \bigoplus_{\lambda\in\mathbb{Z}} F^{\lambda+1} V t^{-\lambda}$. □

Remark 1.14. Using the computation in the above proof, it follows that $s \in F^\lambda V$ if and only if $\overline{s} t^{-\lambda} \in \mathcal{V}$, where $\overline{s}$ is the $\mathbb{G}_m$-invariant section of $\mathcal{V}_{\mathbb{A}^1\setminus 0}$ such that $\overline{s}_1 = s$. To see this note that the injection $H^0(\mathbb{A}^1, \mathcal{V}) \hookrightarrow H^0(\mathbb{A}^1 \setminus 0, \mathcal{V})$ can be written as the composition

$$H^0(\mathbb{A}^1, \mathcal{V}) \xrightarrow{\simeq} \bigoplus_{\lambda\in\mathbb{Z}} F^\lambda V t^{-\lambda} \hookrightarrow \bigoplus_{\lambda\in\mathbb{Z}} V t^{-\lambda} \xrightarrow{\simeq} H^0(\mathbb{A}^1 \setminus 0, \mathcal{V})$$

and $\overline{s}$ corresponds st^0 in the right center module.

Example 1.15. Assume $\mathcal{V}$ is a $\mathbb{G}_m$-linearized vector bundle on $\mathbb{A}^1$ of rank 1. In this case, $V := \mathcal{V}_1$ is a 1-dimensional vector space, and, hence, the induced filtration $F^\bullet$ of V has the property: there exists $\mu \in \mathbb{Z}$ such that

$$F^\lambda V = \begin{cases} V & \text{for } \lambda \leq \mu \\ 0 & \text{for } \lambda > \mu \end{cases}.$$

In this case, we see

$$\mathrm{Rees}(F^\bullet) = \bigoplus_{\lambda\leq\mu} V t^{-\lambda} = V t^{-\mu}[t].$$

We leave the following statement as an exercise to the reader.

Exercise 1.16. If $\mathcal{V}$ is a $\mathbb{G}_m$-linearized vector bundle on $\mathbb{A}^1$, then:

(1) There exists an isomorphism of $\mathbb{G}_m$-linearized vector bundles

$$\mathcal{V} \simeq \mathcal{V}_0 \times \mathbb{A}^1.$$

(2) $\mathcal{V}$ is a direct sum of $\mathbb{G}_m$-linearized line bundles on $\mathbb{A}^1$.

Hint: To prove (1), use the Rees correspondence to translate the statement to a problem concerning either filtrations or torsion free graded $k[t]$-modules. Warning the isomorphism is non-canonical! Use (1) to prove (2).

1.4. Filtrations and test configurations. Using the Rees correspondence, we will prove a relationship between test configurations and filtrations of section rings. To begin, we first introduce a definition.

Definition 1.17 ($\mathbb{Z}$-filtrations). Let $R := \bigoplus_{m\in\mathbb{N}} R_m$ be a graded k-algebra such that each R_m is a finite dimensional k-vector space. A *$\mathbb{Z}$-filtration* $F^\bullet R$ of R is the data of a $\mathbb{Z}$-filtration $F^\bullet R_m$ for each m such that

$$F^\lambda R_m \cdot F^{\lambda'} R_{m'} \subset F^{\lambda+\lambda'} R_{m+m'}$$

for all $\lambda, \lambda \in \mathbb{Z}$ and $m, m' \in \mathbb{N}$.

A filtration $F^\bullet R$ of R is called *finitely generated* if the Rees algebra

$$\mathrm{Rees}(F^\bullet R) := \bigoplus_{m\in\mathbb{N}} \bigoplus_{\lambda\in\mathbb{Z}} R_m t^{-\lambda} \subset R[t,t^{-1}]$$

is finitely generated over k.

Let $(\mathcal{X},\mathcal{L})$ be a test configuration of a polarized pair (X,L). Fix a positive integer $r>0$ such that $r\mathcal{L}$ is a line bundle and consider the graded k-algebra

$$R(X,rL) := \bigoplus_{m\in\mathbb{N}} H^0(X,mrL).$$

To construct a filtration of $R(X,rL)$, we use

(i) $H^0(\mathcal{X},mr\mathcal{L})$ is the space of sections of the $\mathbb{G}_m$-linearized vector bundle $\pi_*(\mathcal{L}^{mr})$ on $\mathbb{A}^1$ and
(ii) there is a canonical isomorphism $H^0(\mathcal{X},mr\mathcal{L})_1 \simeq H^0(X,mrL)$

Note that (ii) holds, since $(\mathcal{X},\mathcal{L})_{\mathbb{A}^1\setminus 0} \simeq (X,L)\times(\mathbb{A}^1\setminus 0)$. Hence, we may use Section 1.3 to define a filtration $F^\bullet_{\mathcal{X},\mathcal{L}}$ of $H^0(X,mrL)$ by

$$F^\lambda_{\mathcal{X},\mathcal{L}} H^0(X,mrL) := \mathrm{image}\left(H^0(\mathcal{X},mr\mathcal{L})_\lambda \to H^0(X,mrL)\right).$$

PROPOSITION 1.18. *The collection of subspaces $F^\bullet_{\mathcal{X},\mathcal{L}}$ is a finitely generated $\mathbb{Z}$-filtration of $R(X,rL)$.*

PROOF. To check $F^\bullet_{\mathcal{X},\mathcal{L}}$ is a $\mathbb{Z}$-filtration of $R(X,rL)$, it remains to show

$$F^\lambda_{\mathcal{X},\mathcal{L}} H^0(X,mrL)\cdot F^{\lambda'}_{\mathcal{X},\mathcal{L}} H^0(X,m'rL) \subset F^{\lambda+\lambda'}_{\mathcal{X},\mathcal{L}} H^0(X,(m+m')rL)$$

for all $\lambda,\lambda'\in\mathbb{Z}$ and $m,m'\in\mathbb{N}$. The latter holds by the definition of $F^\bullet_{\mathcal{X},\mathcal{L}}$ and the inclusion

$$H^0(\mathcal{X},mr\mathcal{L})_\lambda \cdot H^0(\mathcal{X},m'r\mathcal{L})_{\lambda'} \subset H^0(\mathcal{X},(m+m')r\mathcal{L})_{\lambda+\lambda'},$$

which follows from the definition of the weight spaces. Hence, $F^\bullet_{\mathcal{X},\mathcal{L}}$ is a $\mathbb{Z}$-filtration of $R(X,rL)$.

To see the filtration is finitely generated, we use that, by the Rees correspondence, the natural map

$$H^0(\mathcal{X},mr\mathcal{L}) = \bigoplus_{\lambda\in\mathbb{Z}} H^0(\mathcal{X},mr\mathcal{L})_\lambda \xrightarrow{\simeq} \bigoplus_{\lambda\in\mathbb{Z}} F^\lambda H^0(X,mrL)t^{-\lambda}$$

is an isomorphism of $\mathbb{Z}$-graded $k[t]$-modules. Hence, we get an isomorphism

$$\bigoplus_{m\in\mathbb{N}} H^0(\mathcal{X},mr\mathcal{L}) \simeq \bigoplus_{m\in\mathbb{N}}\bigoplus_{\lambda\in\mathbb{Z}} F^\lambda_{\mathcal{X},\mathcal{L}} H^0(X,mrL) = \mathrm{Rees}(\mathcal{F}^\bullet_{\mathcal{X},\mathcal{L}} R(X,rL)).$$

Since $\mathcal{L}$ is relatively ample over $\mathbb{A}^1$, the left hand side of the above expression is a finitely generated $k[t]$-algebra, and, hence, so is the right hand side. Therefore, $F^\bullet_{\mathcal{X},\mathcal{L}}$ is finitely generated. □

THEOREM 1.19. *There is a correspondence between test configurations $(\mathcal{X},\mathcal{L})$ of (X,L) and finitely generated filtrations $F^\bullet$ of $R(X,rL)$ for some $r>0$.*

Proof. Proposition 1.18 shows that a test configuration induces a finitely generated $\mathbb{Z}$-filtration. It remains to start with a finitely generated $\mathbb{Z}$-filtration $F^\bullet$ of $R(X,rL)$ and produce a test configuration. Consider the morphism

$$\mathcal{X} := \operatorname{Proj}\left(\bigoplus_{m\in\mathbb{N}}\bigoplus_{\lambda\in\mathbb{Z}} F^\lambda H^0(X,mrL)t^{-\lambda}\right) \xrightarrow{\pi} \mathbb{A}^1,$$

where the Proj is taken with respect to the $\mathbb{N}$-grading. Since $\operatorname{Rees}(F^\bullet R)$ is a finitely graded $k[t]$-algebra with $m=0$ graded component isomorphic to $k[t]$, the morphism π is projective and $\mathcal{L} := \frac{1}{k}\mathcal{O}_{\mathcal{X}}(k)$ is an ample $\mathbb{Q}$-line bundle for $k \gg 0$.

To see $(\mathcal{X},\mathcal{L}) \to \mathbb{A}^1$ is a test configuration, we first note that the $\mathbb{Z}$-grading inside the Proj induces a $\mathbb{G}_m$-action on $\mathcal{X}$ and a $\mathbb{G}_m$-linearization of $\mathcal{L}$. Next, we compute that the natural map

$$\operatorname{Rees}(F^\bullet R(X,rL)) \otimes_{k[t]} k[t^{\pm 1}] \longrightarrow R(X,rL)[t^{\pm 1}]$$

of $\mathbb{N}\times\mathbb{Z}$-graded $k[t]$-algebras (see the proof of Proposition 1.13.1 for a similar argument). Hence, there is a $\mathbb{G}_m$-equivariant morphism

$$(\mathcal{X},\mathcal{L})_{\mathbb{A}^1\setminus 0} \simeq (X,L)\times(\mathbb{A}^1\setminus 0),$$

where $\mathbb{G}_m$ acts on the right hand side as the product of the trivial action on (X,L) and the standard action on $\mathbb{A}^1\setminus 0$. The latter endows $(\mathcal{X},\mathcal{L})$ with the structure of a test configuration of (X,L). $\square$

Remark 1.20. It is natural to ask if the above correspondence is bijective. This is not quite the case, since two finitely generated $\mathbb{Z}$-filtrations

$$F^\bullet R(X,rL) \quad \text{and} \quad F'^\bullet R(X,r'L)$$

that agree on $H^0(X,mL)$ for sufficiently divisible m, but are not equal for all m, induce the same test configuration. Modding out by this relation gives a bijective correspondence.

As a consequence of the Rees correspondence, we prove a few properties of test configurations.

Proposition 1.21. *Let $(\mathcal{X},\mathcal{L})$ be a test configuration of (X,L).*

(1) If X is reduced, then $\mathcal{X}$ is reduced.
(2) If X is a variety, then $\mathcal{X}$ is a variety.
(3) If X is normal and $\mathcal{X}_0$ is reduced, then $\mathcal{X}$ is normal.

Proof of Proposition 1.21.1-2. By Theorem 1.18 and its proof, $(\mathcal{X},\mathcal{L})$ induces a finitely generated $\mathbb{Z}$-filtration $F^\bullet$ of $R := R(X,rL)$ for some $r>0$ such that

$$\mathcal{X} \simeq \operatorname{Proj}\left(\operatorname{Rees}\left(F^\bullet R\right)\right),$$

Now, assume X is reduced. Then R is reduced, and, using that

$$\operatorname{Rees}(F^\bullet R) \subset R[t,t^{-1}],$$

it follows that $\mathrm{Rees}(F^\bullet R)$ is reduced, and, hence, $\mathcal{X}$ is reduced. Statement (2) can be proven in the same way by replacing the word "reduced" with "integral." $\square$

To prove the third part of the proposition, we use Serre's criterion for normality, which is stated using the following conditions.

Definition 1.22. A Noetherian scheme X is called

(1) R_k if X is regular at codimension $\leq k$ points of X
(2) S_k if $\mathrm{depth}(\mathcal{O}_{X,x}) \geq \min\{k, \dim \mathcal{O}_{X,x}\}$ for each point $x \in X$.

Above, $\mathrm{depth}(\mathcal{O}_{X,x})$ denotes the longest length of a regular sequence of $\mathcal{O}_{X,x}$

Proposition 1.23. *Let X be a Noetherian scheme.*

(1) [**SPA22**, Lemma 031R] *X is reduce if and only if it is R_0 and S_1.*
(2) [**SPA22**, Lemma 031S] *X is normal if and only if it is R_1 and S_2.*
(3) [**SPA22**, Definiton 00N3] *X is CM if and only if it is S_k for all k.*

Proof of Propositon 1.21.3. Since X is normal and $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{A}^1 \setminus 0)$, $\mathcal{X} \setminus \mathcal{X}_0$ is normal. Note that $\mathcal{X}_0$ is R_0 and S_1, by our assumption, and t is a non-zero divisor of $\mathcal{O}_{\mathcal{X}}$, by the flatness of $\mathcal{X} \to \mathbb{A}^1$. Thus, [**SPA22**, Tag 00NU] and [**SPA22**, Tag 090R] imply $\mathcal{X}$ is R_1 and S_2 at points in $\mathcal{X}_0$. Hence, $\mathcal{X}$ is normal in a neighborhood of $\mathcal{X}_0$. $\square$

Warning 1.24. The assumption in 1.21 that $\mathcal{X}_0$ is reduced is necessary. Indeed, [**LX14**, Example 4] gives an example of a test configuration of $\mathbb{P}^1$ that is not normal. In the example, the test configuration is the degeneration of a twisted cubic to a plane cubic with an embedded point.

2. Futaki invariant

The Futaki invariant was first defined analytically as a linear functional on the lie algebra of the space of holomorphic vector fields of a smooth Fano variety [**Fut83**]. The latter definition was generalized by Tian and Diang for singular Fano varieties [**DT92**]. We will discuss Donaldson's version of the Futaki invariant, which is defined algebraically for any test configuration [**Don02**].

2.1. Weights. The Futaki invariant of a test configuration $(\mathcal{X}, \mathcal{L})$ is defined in terms of the $\mathbb{G}_m$-action on

$$H^0(\mathcal{X}_0, m\mathcal{L}_0)$$

as $m \to \infty$. The definition uses the following notion.

Definition 2.1. The *weight* of a $\mathbb{G}_m$-action on a finite dimensional vector space V is defined as

$$\mathrm{wt}(V) = \sum_{\lambda \in \mathbb{Z}} \lambda \dim(V_\lambda),$$

where $V = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda$ is the weight decomposition of V, which means $V_\lambda := \{v \in V \mid \xi \cdot v = \xi^\lambda v \text{ for all } \xi \in \mathbb{G}_m(k)\}$.

REMARK 2.2. If $\mathbb{G}_m$ acts on a finite dimensional vector space V of dimension N, then

$$\mathrm{wt}(V) = \mathrm{wt}(\det(V)),$$

where $\det V := \bigwedge^N V$, which has an induced $\mathbb{G}_m$-action. To see this, choose a basis $s_1, \ldots, s_N$ for V and $\lambda_1, \ldots, \lambda_N \in \mathbb{Z}$ such that $\xi \cdot s_i = \xi^{\lambda_i} s_i$. Since

$$\xi \cdot s_1 \wedge \ldots \wedge s_N = \xi^{\sum_i \lambda_i} s_1 \wedge \ldots \wedge s_N$$

for each $\xi \in \mathbb{G}_m(k)$, we see

$$\mathrm{wt}(\det(V)) = \sum_{i=1}^{N} \lambda_i = \mathrm{wt}(V).$$

2.2. Donaldson's Futaki invariant. Let $(\mathcal{X}, \mathcal{L})$ be a test configuration of (X, L) and $n := \dim(X)$. For each positive integer m such that $m\mathcal{L}$ is a line bundle, set

$$N_m := \dim H^0(\mathcal{X}_0, m\mathcal{L}_0) \quad \text{and} \quad w_m := \mathrm{wt} H^0(\mathcal{X}_0, m\mathcal{L}_0).$$

For $m > 0$ sufficiently divisible, N_m and w_m agree with polynomial functions. In particular, by Proposition 2.7 proven below, there exists rational numbers a_i and b_i such that

$$N_m := a_0 m^n + a_1 m^{n-1} + \ldots + a_n$$
$$w_m := b_0 m^{n+1} + b_1 m^n + \ldots + b_{n+1}$$

for $m > 0$ sufficiently divisible. Hence, there exist rational numbers F_i such that

$$\frac{w_m}{mN_m} = F_0 + F_1 m^{-1} + F_2 m^{-2} + \ldots$$

for $m > 0$ sufficiently divisible.

DEFINITION 2.3 (Futaki invariant). The *Futaki* invariant of $(\mathcal{X}, \mathcal{L})$ is

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) := -2F_1.$$

By solving for F_1 in terms of the a_i and b_i, it follows that $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \frac{2(b_0 a_1 - b_1 a_0)}{a_0^2}$.

2.3. Compactification. In order to express the Futaki invariant as an intersection number, we will need to work with compact spaces.

DEFINITION 2.4. The *compactification* of a test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L), which is denoted by

$$\overline{\pi} : (\overline{\mathcal{X}}, \overline{\mathcal{L}}) \to \mathbb{P}^1,$$

is the gluing the two equivariant families $(\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ and $(X, L) \times (\mathbb{P}^1 \setminus 0) \to (\mathbb{P}^1 \setminus 0)$ along their isomorphic open subsets

$$\begin{tikzcd}
(\mathcal{X}, \mathcal{L}) & (X, L) \times (\mathbb{P}^1 \setminus 0) \\
(X, L) \times (\mathbb{A}^1 \setminus 0) \arrow[u, hook] \arrow[r, equal] & (X, L) \times (\mathbb{A}^1 \setminus 0) \arrow[u, hook]
\end{tikzcd}.$$

In the above gluing, the $\mathbb{G}_m$-action on $(X, L) \times (\mathbb{P}^1 \setminus 0)$ is the product of the trivial action on (X, L) and the standard action on $\mathbb{P}^1 \setminus 0$.

Observe that

- $\overline{\pi} : \overline{\mathcal{X}} \to \mathbb{P}^1$ is a $\mathbb{G}_m$-equivariant flat proper morphism of schemes,
- $\overline{\mathcal{L}}$ is a $\mathbb{G}_m$-linearized $\overline{\pi}$-ample $\mathbb{Q}$-line bundle on $\overline{\mathcal{X}}$, and
- the fiber over ∞, which is denoted by $(\overline{\mathcal{X}}_\infty, \overline{\mathcal{L}}_\infty)$, is isomorphic to (X, L) with the trival $\mathbb{G}_m$-action.

EXAMPLE 2.5. Let $\mathcal{X} := \mathbb{P}^1 \times \mathbb{A}^1$ be the product test configuration induced by the $\mathbb{G}_m$-action on $\mathbb{P}^1$ given by

$$t \cdot [x : y] = [t^d x : y]$$

for some integer d. Then the compactification of $\mathcal{X} \to \mathbb{A}^1$ is

$$\overline{\mathcal{X}} := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)) \to \mathbb{P}^1,$$

where $\overline{\mathcal{X}}$ is a Hirzebruch surface.

2.4. Intersection formula. In this section, we prove Odaka and Wang's formula for the Futaki invariant in terms of intersection numbers [**Oda13**, **Wan12**].

THEOREM 2.6 ([**Wan12**, **Oda13**]). *If $(\mathcal{X}, \mathcal{L})$ is a test configuration of (X, L) and $\mathcal{X}$ is normal, then*

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \frac{\overline{\mathcal{L}}^n \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1}}{V} + \overline{S} \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)V},$$

where $V = L^n$ and $\overline{S} = nV^{-1}(-K_X \cdot L^{n-1})$.

The theorem follows from the following more detailed proposition.

PROPOSITION 2.7. *If $(\mathcal{X}, \mathcal{L})$ is a test configuration of (X, L) and $n = \dim(X)$, then there exists rational numbers a_i and b_i such that*

$$N_m := \dim H^0(\mathcal{X}_0, m\mathcal{L}_0) = a_0 m^n + a_1 m^{n-1} + \ldots + a_n$$
$$w_m := \mathrm{wt}(H^0(\mathcal{X}_0, m\mathcal{L}_0) = b_0 m^{n+1} + b_1 m^n + \ldots + b_{n+1}$$

for all $m > 0$ sufficiently divisible. Furthermore,

$$a_0 = \frac{L^n}{n!} \quad \text{and} \quad b_0 = \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)!}.$$

Additionally, if $\mathcal{X}$ is normal, then

$$a_1 = -\frac{L^{n-1} \cdot K_X}{2(n-1)!} \quad \text{and} \quad b_1 = -\frac{\overline{\mathcal{L}}^n \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1}}{2n!}$$

PROOF. If m is a sufficiently divisible positive integer, then Serre vanishing implies

$$H^i(\overline{\mathcal{X}}_t, m\overline{\mathcal{L}}_t)) = 0$$

for all $t \in \mathbb{P}^1$. For such m, [**Har77**, Theorem 12.11] implies

(i) $R^i\overline{\pi}_*\mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}}) = 0$ for all $i > 0$,
(ii) $\overline{\pi}_*\mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})$ is a vector bundle, and
(iii) and the natural map $\overline{\pi}_*\mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}}) \otimes k(t) \to H^0(\mathcal{X}_t, m\mathcal{L}_t)$ is an isomorphism for all $t \in \mathbb{P}^1$.

Below, we always assume $m > 0$ is sufficiently divisible so that these statements hold.

Now, observe that

$$N_m := H^0(\mathcal{X}_0, m\mathcal{L}_0) = H^0(\mathcal{X}_1, m\mathcal{L}_1) = H^0(X, mL),$$

where the second equality holds by (ii) and (iii). Hence, Theorem 1.7 implies N_m agrees with a polynomial of degree $\leq n$ and the desired formulas for a_0 and a_1 hold.

It remains to analyze the weight function w_m. By (ii), $\det(\overline{\pi}_*\mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}}))$ is a $\mathbb{G}_m$-linearized line bundle on $\mathbb{P}^1$. By (iii),

$$\operatorname{wt}(\det(\overline{\pi}_*\mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}}))_0) = \operatorname{wt}\det H^0(\mathcal{X}_0, m\mathcal{L}_0) = w_m.$$

Since $\mathbb{G}_m$ acts trivially on $(\overline{\mathcal{X}}, \overline{\mathcal{L}})_\infty$, we also have

$$\operatorname{wt}(\det(\overline{\pi}_*\mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}}))_\infty) = 0.$$

Now, Proposition 3.7 implies

$$\det(\overline{\pi}_*\mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) \simeq \mathcal{O}_{\mathbb{P}^1}(w_m).$$

The latter isomorphism is the key to relating the weight w_m to an intersection number! We now compute

$$\begin{aligned} w_m = \deg(\mathcal{O}_{\mathbb{P}^1}(w_m)) = \deg(\overline{\pi}_*\mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) &= \chi(\mathbb{P}^1, (\overline{\pi}_*\mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) - \operatorname{rk}((\overline{\pi}_*\mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}}) \\ &= \chi(\mathbb{P}^1, (\overline{\pi}_*\mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) - N_m \\ &= \chi(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) - N_m \end{aligned}$$

where the third equality is by the Riemann-Roch formula for vector bundles on curves, which can be deduced from [**Har77**, Theorem IV.1.3], the fourth uses that

$$\operatorname{rk}(\overline{\pi}_*\mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) = \dim H^0(\mathcal{X}_0, m\mathcal{L}_0) = N_m$$

by (ii) and (iii), and the fifth is by (i) combined with [**Har77**, Exercise III.8.1]. Since $\chi(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}}))$ and N_m agree with polynomials of degree $\leq n+1$ by Theorem 1.7, w_m agrees with a polynomial as well.

The final step in the proof is to compute b_0 and b_1. Theorem 1.7 implies

$$\begin{aligned} w_m &= \chi(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) - N_m \\ &= \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)!} m^{n+1} + \text{ lower order terms}, \end{aligned}$$

Hence, the formula for b_0 holds. Additionally, if $\mathcal{X}$ is normal, then

$$w_m = \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)!} m^{n+1} - \frac{\overline{\mathcal{L}}^n \cdot K_{\overline{\mathcal{X}}} + 2L^n}{2n!} m^n + \text{ lower order terms.}$$

Using that
$$2L^n = 2\overline{\mathcal{L}}^n \cdot \mathcal{O}_{\overline{\mathcal{X}}}(\mathcal{X}_1) = \overline{\mathcal{L}}^n \cdot \pi^*\mathcal{O}_{\mathbb{P}^1}(2) = -\overline{\mathcal{L}}^n \cdot \overline{\pi}^* K_{\mathbb{P}^2},$$
we deduce the desired formula for b_1. □

PROOF OF THEOREM 2.6. The theorem is an immediate consequence of Proposition 2.7 and the formula
$$\operatorname{Fut}(\mathcal{X}, \mathcal{L}) = \frac{2(b_0 a_1 - b_1 a_0)}{a_0^2}.$$
□

2.5. Normalization of test configurations. A powerful tool for understanding K-stability is the intersection formula for the Futaki invariant. Since the formula only holds and makes sense for normal test configurations, it will be useful to take the normalization of a test configuration.

DEFINITION 2.8 (Normalization). Let $(\mathcal{X}, \mathcal{L})$ be a test configuration of a polarized scheme (X, L), with X a normal variety. The *normalization* of $(\mathcal{X}, \mathcal{L})$ is
$$(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}}),$$
where $\nu : \widetilde{\mathcal{X}} \to \mathcal{X}$ is the normalization morphism and $\widetilde{\mathcal{L}} := \nu^*\mathcal{L}$.

Observe that $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ is naturally a test configuration of (X, L). Indeed, the composition
$$\widetilde{\mathcal{X}} \to \mathcal{X} \to \mathbb{A}^1$$
is flat by [Har77, III.9.7] and proper. Additionally, using the universal property of the normalization morphism, there exists a unique morphism $\widetilde{\sigma} : \mathbb{G}_m \times \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}$, which makes the diagram commute
$$\begin{tikzcd} \mathbb{G}_m \times \widetilde{\mathcal{X}} \arrow[r, dashed, "\exists!"] \arrow[d, "\mathrm{id}\times\nu"] & \widetilde{X} \arrow[d, "\nu"] \\ \mathbb{G}_m \times \mathcal{X} \arrow[r] & \mathcal{X} \end{tikzcd}.$$
Furthermore, $\widetilde{\sigma}$ is a $\mathbb{G}_m$-action on $\widetilde{\mathcal{X}}$, since the diagrams in the definition of a $\mathbb{G}_m$-action commute over an open set and hence, everywhere. Next, since $\mathcal{X}|_{\mathbb{A}^1 \setminus 0} \simeq X \times (\mathbb{A}^1 \setminus 0)$ is normal, there is an induced isomorphism
$$\widetilde{\mathcal{X}}_{\mathbb{A}^1\setminus 0} \simeq X \times (\mathbb{A}^1 \setminus 0).$$
Finally, $\widetilde{\mathcal{L}}$ admits a natural $\mathbb{G}_m$-linearization. It is ample over $\mathbb{A}^1$, since $\mathcal{L}$ is ample over $\mathbb{A}^1$ and ν is finite.

REMARK 2.9. More generally, this construction makes sense without even assuming X is normal. In the general setting, the normalization $\widetilde{X} \to X$ is defined to be the normalization of the reduction X_{red} and $\widetilde{L}$ denotes the pullback of L. If $(\mathcal{X}, \mathcal{L})$ is a test configuration of (X, L), then its normalization $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ is a test configuration of $(\widetilde{X}, \widetilde{L})$.

The following proposition shows that when testing the positivity of the Futaki invariant, it suffices to consider normal test configurations.

PROPOSITION 2.10. *Let (X, L) be a polarized scheme with X normal. If $(\mathcal{X}, \mathcal{L})$ is a test configuration of (X, L), then*

$$\mathrm{Fut}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}}) \leq \mathrm{Fut}(\mathcal{X}, \mathcal{L}).$$

PROOF. To simplify notation in the proof, let $(\mathcal{X}', \mathcal{L}')$ denote the normalization of $(\mathcal{X}, \mathcal{L})$. Let

$$w'_m := \mathrm{wt}(H^0(\mathcal{X}'_0, m\mathcal{L}'_0) \quad \text{ and } \quad w_m := \mathrm{wt}(H^0(\mathcal{X}_0, m\mathcal{L}_0)).$$

Additionally, set

$$N'_m := \dim(H^0(\mathcal{X}'_0, m\mathcal{L}'_0) \quad \text{ and } \quad N_m := \dim(H^0(\mathcal{X}_0, m\mathcal{L}_0)).$$

Note that for $m > 0$ sufficiently divisible

$$N'_m = H^0(X, mL) = N_m.$$

It remains to compare w'_m and w_m.

As observed in the proof of Proposition 2.7, for any test configuration $(\mathcal{X}, \mathcal{L})$

$$w_m = \chi(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) - N_m$$

for all $m > 0$ sufficiently divisible. Hence, we consider the short exact sequence

$$0 \to \mathcal{O}_{\overline{\mathcal{X}}} \to \overline{\nu}_* \mathcal{O}_{\overline{\mathcal{X}}'} \to \mathcal{F} \to 0,$$

where $\mathcal{F} := (\overline{\nu}_* \mathcal{O}_{\overline{\mathcal{X}}'})/\mathcal{O}_{\overline{\mathcal{X}}}$. Tensoring by $\cdot \otimes \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})$ and using the projection formula, gives

$$0 \to \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}}) \to \overline{\nu}_*(\mathcal{O}_{\overline{\mathcal{X}}'}(m\overline{\mathcal{L}}')) \to \mathcal{F} \otimes \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}}) \to 0$$

Thus, for $m > 0$ sufficiently divisible,

$$\begin{aligned} \chi(\overline{\mathcal{X}}', \mathcal{O}_{\overline{\mathcal{X}}'}(m\overline{\mathcal{L}}')) &= \chi(\overline{\mathcal{X}}, \overline{\nu}_* \mathcal{O}_{\overline{\mathcal{X}}'}(m\overline{\mathcal{L}}')) \\ &= \chi(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) + \chi(\overline{\mathcal{X}}, \mathcal{F} \otimes \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) \\ &= \chi(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) + H^0(\overline{\mathcal{X}}, \mathcal{F} \otimes \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})). \end{aligned}$$

Above, the first equality holds by the fact that $\overline{\nu}$ is affine and [**Har77**, Exercise III.8.2]. For the third equality, note that $\mathcal{L}|_{\mathcal{X}_0}$ is ample and $\mathrm{Supp}(\mathcal{F}) \subset \mathcal{X}_0$, so $H^i(\overline{\mathcal{X}}, \mathcal{F} \otimes \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) = 0$ for all $i > 0$ and sufficiently large $m > 0$ by Serre vanishing. Therefore,

$$w'_m = w_m + H^0(\overline{\mathcal{X}}, \mathcal{F} \otimes \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})).$$

Since $\dim(\mathrm{Supp}(\mathcal{F})) \leq n := \dim(X)$, [**Kle66**, Section 1] implies

$$H^0(\overline{\mathcal{X}}, \mathcal{F} \otimes \mathcal{O}_{\overline{\mathcal{X}}}(m\overline{\mathcal{L}})) = O(m^n)$$

where $n = \dim(X)$. Therefore,

$$w'_m \geq w_m \quad \text{and} \quad w'_m = w_m + O(m^n)$$

for $m > 0$ sufficiently divisible, which implies $\mathrm{Fut}(\mathcal{X}', \mathcal{L}') \leq \mathrm{Fut}(\mathcal{X}, \mathcal{L})$ as desired. □

The following proposition gives a characterization of normal trivial test configurations.

Proposition 2.11. *Let (X, L) be a polarized variety with X normal and $(\mathcal{X}, \mathcal{L})$ a normal test configuration A normal test configuration $(\mathcal{X}, \mathcal{L})$ is trivial if and only if the birational map*

$$\phi : \mathcal{X} \dashrightarrow X \times \mathbb{A}^1$$

is an isomorphism in codimension 1.

Proof. The forward implication is trivial. For the reverse implication, assume ϕ is an isomorphism in codimension 1. Hence, there exists a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, dashed, "\phi"] & X \times \mathbb{A}^1 \\ U \arrow[u, hook] \arrow[r, "\psi"] & V \arrow[u, hook] \end{tikzcd}$$

such that ψ is an isomorphism and the vertical arrows are open embedding with complement codimension at least 2.

Since $(\mathcal{X}, \mathcal{L})$ and $(X_{\mathbb{A}^1}, L_{\mathbb{A}^1})$ are isomorphic over $(\mathbb{A}^1 \setminus 0)$,

$$L_{\mathbb{A}^1}|_V \otimes \psi_*(\mathcal{L}^\vee|_U)$$

is trivial over $V \setminus V_0$. Hence,

$$L_{\mathbb{A}^1}|_V \otimes \psi_*(\mathcal{L}^\vee|_U) = \mathcal{O}_X(d(X \times 0))|_V$$

for some integer d, which implies

$$L_{\mathbb{A}^1}(d(X \times 0))|_V \simeq \psi_*(\mathcal{L}|_U).$$

Finally, observe that we have natural maps

$$\mathcal{X} \simeq \mathrm{Proj} \bigoplus_{m \in \mathbb{N}} H^0(\mathcal{X}, m\mathcal{L}) \simeq \mathrm{Proj} \bigoplus_{m \in \mathbb{N}} H^0(X_{\mathbb{A}^1}, m(L_{\mathbb{A}^1} + d(X \times 0))) \simeq X_{\mathbb{A}^1},$$

where the second isomorphism is a consequence of the algebraic Hartog's Lemma [**Har77**, Proposition 6.3A]. □

2.6. Definition of K-stability.

Definition 2.12. Let X be a proper normal variety and L an ample line bundle on X. The pair (X, L) is:

(1) *K-semistable* if and only if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$ for all normal test configurations $(\mathcal{X}, \mathcal{L})$ of (X, L),
(2) *K-polystable* if and only if (X, L) is K-semistable and $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = 0$ only when $(\mathcal{X}, \mathcal{L})$ is a product, and

(3) *K-stable* if and only if (X, L) is K-semistable and $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = 0$ only when $(\mathcal{X}, \mathcal{L})$ is trivial.

The above definition uses the notion of product and trivial test configurations. Recall, a test configuration $(\mathcal{X}, \mathcal{L})$ is a *product* if there is $\mathbb{G}_m$-equivariant isomorphism over $\mathbb{A}^1$

$$\mathcal{X} \simeq X \times \mathbb{A}^1,$$

where the $\mathbb{G}_m$-action on $X \times \mathbb{A}^1$ is the product of a $\mathbb{G}_m$-action on X and the standard action on $\mathbb{A}^1$. It is *trivial* if the $\mathbb{G}_m$-action on $X \times \mathbb{A}^1$ is the product of the trivial action on X and the standard action on $\mathbb{A}^1$.

Remark 2.13. We consider the relationship between the three stability notions.

(1) It follows from the definition that

$$\text{K-stability} \implies \text{K-polystability} \implies \text{K-semistability}$$

(2) If there exists a non-trivial $\mathbb{G}_m$-action on X and a $\mathbb{G}_m$-linearization of L, then (X, L) is not K-stable. Indeed, a non-trivial $\mathbb{G}_m$-action on (X, L) and its inverse induce non-trivial product test configurations $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}', \mathcal{L}')$ with

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) + \mathrm{Fut}(\mathcal{X}', \mathcal{L}') = 0.$$

Hence, at least one of the Futaki invariants is non-positive.

(3) If X is a smooth Fano variety over $\mathbb{C}$, then:
  (a) $(X, -K_X)$ is K-polystable if and only if X admits a Kähler-Einstein metric [**CDS15**, **Tia15**];
  (b) $(X, -K_X)$ is K-stable if and only if X admits a Kähler-Einstein metric and has discrete automorphism group.
  (c) $(X, -K_X)$ is K-semistable if and only if there exists a test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -K_X)$ such that $\mathcal{X}_0$ is a possibly singular Fano variety with Kähler-Einstein metric [**Li17a**].

Similar results are known to hold when X is singular by [**Li22**, **LXZ22**].

Remark 2.14. In Definition 2.12, we only consider normal test configurations.

(1) In the definition of K-semistablity, it would be equivalent to consider all test configuration by Proposition 2.10.

(2) In the definition of K-stability and K-polystability, it is essential that we only consider normal test configurations. Indeed, [**LX14**, Example 14] gives an example of a test configuration $(\mathcal{X}, \mathcal{L})$ of $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ with an embedding $\mathcal{X} \subset \mathbb{P}^2 \times \mathbb{A}^1$ such that $\mathcal{X}_1 \subset \mathbb{P}^2$ is a twisted cubic and $\mathcal{X}_0 \subset \mathbb{P}^2$ is a plane cubic with an embedded point. In this example,

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \mathrm{Fut}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}}) = 0$$

and $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ is a trivial test configuration. Hence, if we considered all test configurations in Definiton 2.12, then $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ would not be K-polystable. In fact, due to similar examples [**Oda15b**, Proposition 3.5], almost nothing could be K-stable or K-polystable if we required $\operatorname{Fut}(\mathcal{X}, \mathcal{L}) > 0$ for all (not necessarily normal) non-product test configurations!

REMARK 2.15. In Defnition 2.12, we only defined K-stability in the case when X is normal. In future sections, we will primarily be interested in this case.

More generally, one can define K-(semi/poly)stability without the normality assumption on X. For this, one defines a test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L) to be *almost trivial* (resp., *almost a product*) if its normalization $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ is a trivial (resp., product) test configuration of $(\widetilde{X}, \widetilde{L})$. Then in Definition 2.12, we consider all test configurations (not just normal ones) and require that $\operatorname{Fut}(\mathcal{X}, \mathcal{L}) = 0$ only when $(\mathcal{X}, \mathcal{L})$ is almost a product (resp., almost trivial).

CHAPTER 3

Odaka's Theorems

In this chapter, we discuss Odaka's theorems on K-stability. The first solves the problem of which Calabi-Yau and canonically polarized varieties are K-stable. The second shows that K-semistability imposes conditions on singularities.

These results and their proof showed the first connections between K-stability and the Minimal Model Program. As we will see in future chapter, connections between these two fields will be the key to understanding the K-stability of Fano varieties.

Conventions: Throughout, all schemes are defined over an algebraically closed field k of characteristic 0.

1. Calabi-Yau and canonically polarized varieties

The following theorem determines which Calabi-Yau varieties and canonically polarized varieties are K-stable.

Theorem 1.1. *Let X be a normal variety and L an ample $\mathbb{Q}$-line bundle on X.*

(1) Assume $K_X \sim_{\mathbb{Q}} 0$. Then

$$\begin{aligned} X \text{ is klt} \quad &\iff \quad (X,L) \text{ is K-stable} \\ X \text{ is lc} \quad &\iff \quad (X,L) \text{ is K-semistable} \end{aligned}$$

(2) Assume $K_X = L$. Then

$$X \text{ is lc} \quad \iff \quad (X,L) \text{ is K-stable} \quad \iff \quad (X,L) \text{ is K-semistable.}$$

The result was proven in [**Oda12**, **Oda13**]. See also [**BHJ17**, Section 9.1] for a similar argument with different exposition.

Example 1.2. The theorem immediately implies that the following polarized curves are K-stable.

(1) (E,L), where E is an elliptic curve and L an ample line bundle on E.
(2) (C,K_C), where C is a smooth curve of genus $g \geq 2$.

Remark 1.3 (Canonically polarized varieties with non-canonical polarizations). In Theorem 1.1.2, it is essential that $K_X = L$. Indeed, by [**Ros06**], for each $g \geq 5$, there exists a polarized scheme

$$(C \times C, L),$$

where C is a smooth genus g curve and L an ample line bundle on $C \times C$, that is not K-semistable. Note that in this example $K_{C\times C} = p_1^* K_C \otimes p_2^* K_C$, which is ample.

Remark 1.4 (Relation to complex geometry). In the case when the ground field $k = \mathbb{C}$, the result mirrors existence statements for Kähler-Einstein metrics on possibly singular Calabi-Yau and canonically polarized varieties [EGZ09, BG14].

In this section, we will prove the forward implication of Theorem 1.1. The argument relies on an intersection number computation and the definitions of klt and lc singularities. The proof even works in characteristic p.

The proof of the reverse implication of Theorem 1.1 requires machinery Minimal Model Program. Its proof will be discussed in Section 2, which focuses on a more general result.

1.1. A modified intersection formula. A key ingredient in the proof of Odaka's theorems is a modified version of the intersection formula for test configurations.

Let $(\mathcal{X}, \mathcal{L})$ be a normal test configuration of a polarized variety (X, L). Consider the diagram

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \overline{\mathcal{X}} \arrow[rr, dashed] & & X \times \mathbb{P}^1 \end{tikzcd},$$

where $\mathcal{Y}$ is the main component of the normalization of the graph of $\overline{\mathcal{X}} \dashrightarrow X \times \mathbb{P}^1$ and f, g are the natural morphisms to $\overline{\mathcal{X}}$ and $X \times \mathbb{P}^1$.

Proposition 1.5. *With the above setup,*

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \frac{\overline{\mathcal{L}}^n \cdot f_*(K_{\mathcal{Y}/X\times\mathbb{P}^1} + g^*p_1^*K_X)}{V} + \overline{S}\frac{\overline{\mathcal{L}}^{n+1}}{(n+1)V},$$

where $\overline{S} := nV^{-1}(-K_X \cdot L^n)$.

The advantage of the above formula over the original intersection formula in Theorem 2.6 is that involves the relative canonical divisor $K_{\mathcal{Y}/X\times\mathbb{P}^1}$, which is related to the singularities of X.

Proof. Recall, Theorem 2.6 states that

$$\frac{\overline{\mathcal{L}}^n \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1}}{V} + \overline{S}\frac{\overline{\mathcal{L}}^{n+1}}{(n+1)V}.$$

Hence, it suffices to show that

$$f_*(K_{\mathcal{Y}/X\times\mathbb{P}^1} + g^*p_1^*K_X) = K_{\mathcal{X}/\mathbb{P}^1}.$$

To prove the latter observe that

$$\begin{aligned} K_{\mathcal{Y}/X\times\mathbb{P}^1} + g^*p_1^*K_X &= K_{\mathcal{Y}} - g^*(K_{X\times\mathbb{P}^1} - p_1^*K_X) \\ &= K_{\mathcal{Y}} - g^*p_2^*(K_{\mathbb{P}^1}) \\ &= K_{\mathcal{Y}} - f^*\pi^*(K_{\mathbb{P}^1}), \end{aligned}$$

where the second equality uses that $K_{X\times\mathbb{P}^1} = p_1^*K_X + p_2^*K_{\mathbb{P}^1}$, which can be deduced from [**Har77**, Exercise II.8.3.b] applied on the smooth locus of $X\times\mathbb{P}^1$. Hence,

$$f_*(K_{\mathcal{Y}/X\times\mathbb{P}^1} + g^*p_1^*K_X) = f_*K_{\mathcal{Y}} - f_*f^*\pi^*(K_{\mathbb{P}^1}) = K_{\overline{\mathcal{X}}} - \overline{\pi}^*K_{\mathbb{P}^1} = K_{\mathcal{X}/\mathbb{P}^1}$$

as desired. □

Proposition 1.6. *Keep the above notation.*

(1) If X is lc, then $K_{\mathcal{Y}/X\times\mathbb{P}^1}$ is effective.
(2) If X is klt, then $K_{\mathcal{Y}/X\times\mathbb{P}^1}$ is effective and has support equal to $\mathcal{Y}_0$.

Proof. First, assume X is lc. We claim that $(X\times\mathbb{P}^1, X\times 0)$ is lc. Indeed, this follows from inversion of adjunction [**KM98**, Theorem 5.50] or the following simple argument. Fix a log resolution $f : Y \to X$ of X. Since X is lc, $K_{Y/X} = \sum_{i=1}^r a_iE_i$ for some prime divisors $E_i \subset Y$ and $a_i \geq -1$. Now, note that $f_{\mathbb{P}^1} : Y\times\mathbb{P}^1 \to X\times\mathbb{P}^1$ is a log resolution of $(X\times\mathbb{P}^1, X\times 0)$ and

$$K_{Y\times\mathbb{P}^1/X\times\mathbb{P}^1} - f_{\mathbb{P}^1}^*(X\times 0) = \sum_{i=1}^r a_iE_i - Y\times 0,$$

which has coefficients ≥ -1. Hence, $(X\times\mathbb{P}^1, X\times 0)$ is lc. The latter implies

$$K_{\mathcal{Y}/X\times\mathbb{P}^1} - g^*(X\times 0) = K_{\mathcal{Y}/X\times\mathbb{P}^1} - \mathcal{Y}_0$$

has coefficients ≥ -1. Since $\mathrm{Supp}(K_{\mathcal{Y}/X\times\mathbb{P}^1}) \subset \mathrm{Exc}(g) = \mathcal{Y}_{0,\mathrm{red}}$, it follows that $K_{\mathcal{Y}/X\times\mathbb{P}^1}$ has coefficients ≥ 0, which proves (1). A similar argument proves (2). □

1.2. Proof of Theorem 1.1. As a consequence of the modified intersection formula in Section 1.1, we can now prove the forward implication of Theorem 1.1.

Proof of the forward implications of Theorem 1.1. Let $(\mathcal{X},\mathcal{L})$ be a nontrivial normal test configuration of (X,L). Assume $K_X \sim_{\mathbb{Q}} 0$ and X is lc. Then Theorem 1.5 implies

$$\mathrm{Fut}(\mathcal{X},\mathcal{L}) = \frac{\overline{\mathcal{L}}^n\cdot f_*(K_{\mathcal{Y}/X\times\mathbb{P}^1})}{V}$$

and $f_*K_{\mathcal{Y}/X\times\mathbb{P}^1}$ is effective by Proposition 1.6. Since $\mathrm{Supp}(f_*K_{\mathcal{Y}/X\times\mathbb{P}^1}) \subset \mathcal{X}_{0,\mathrm{red}}$ and $\mathcal{L}|_{\mathcal{Y}_{0,\mathrm{red}}}$ is ample, it follows that $\mathrm{Fut}(\mathcal{X},\mathcal{L}) \geq 0$. Hence, (X,L) is K-semistable.

If $K_X\sim_{\mathbb{Q}} 0$ and X is klt, the same argument applies, but Lemma 2.11 and Proposition 1.6 imply $f_*K_{\mathcal{Y}/X\times\mathbb{P}^1}$ is effective and non-zero. Thus, the above argument shows $\mathrm{Fut}(\mathcal{X},\mathcal{L}) > 0$. Hence, (X,L) is K-stable.

Finally, assume $K_X = L$ and X is lc. In this case, the formula in Theorem 1.5 gives

$$\mathrm{Fut}(\mathcal{X},\mathcal{L}) = \frac{\overline{\mathcal{L}}^n\cdot f_*(K_{\mathcal{Y}/X\times\mathbb{P}^1})}{V} + \frac{(f^*\overline{\mathcal{L}}^n\cdot g^*L_{\mathbb{A}^1}) - \frac{n}{n+1}\overline{\mathcal{L}}^{n+1}}{(n+1)V}.$$

As argued above, $\frac{\overline{\mathcal{L}}^n \cdot f_*(K_{\mathcal{Y}/X\times\mathbb{P}^1})}{V} \geq 0$. Additionally, [**BHJ17**, Proposition 7.8] shows[1]

$$\frac{(f^*\overline{\mathcal{L}}^n \cdot g^* L_{\mathbb{A}^1}) - \frac{n}{n+1}\overline{\mathcal{L}}^{n+1}}{(n+1)V} > 0.$$

(In fact this inequality holds for all normal test configurations without the assumption that $K_X = L$. The argument relies on the Hodge Index Theorem.) □

2. K-semistability implies log canonical

The following theorem shows a suprising connection between K-stability and singularities in the Minimal Model Program.

THEOREM 2.1. [**Oda13**] *Let X be a normal variety with K_X $\mathbb{Q}$-Cartier and L an ample $\mathbb{Q}$-line bundle on X. If (X, L) is K-semistable, then X is lc.*

REMARK 2.2 (Singularities and stability). It is natural that a "stability" notion should impose conditions on singularities.

(1) Stable curves by definition have at worst nodal singularities [**DM69**].
(2) If a polarized variety (X, L) is asymptotically Chow semistable, then
$$\mathrm{mult}(x, X) \leq (\dim(X) + 1)!$$
at each $x \in X$, where $\mathrm{mult}(x, X)$ denotes the Hilbert-Samuel multiplicity of $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ [**Mum77**, Proposition 3.12]. Here, asymptotic Chow stability is a notion that arises from GIT and imple K-semistability.

The proof of Theorem 2.1 is in fact related to the argument in [**Mum77**].

REMARK 2.3 (Klt singularities). In [**Oda13**], Odaka also proves that X has klt singularities in two important cases.

(1) Fano case: Assume (X, L) in Theorem 2.1 additionally satisfies $L = -K_X$. If (X, L) is K-semistable, then X is klt.
(2) Calabi-Yau case: Assume (X, L) in Theorem 2.1 additionally satisfies $K_X \sim_{\mathbb{Q}} 0$. If (X, L) is K-semistable, then X is klt.

REMARK 2.4 (Non-normal case). A version of Theorem 2.1 holds without the assumption that X is normal. In particular, if we replace "normal" with "S_2 Gorenstein at codimension 1 points", then [**Oda13**] proves: if (X, L) K-semistabile, then X is slc.

Slc singularities are a non-normal version of lc singularties. In dimension 1, slc singularities are either smooth points or nodes. In higher dimensions, slc singularities appear naturally when compactifying the moduli space of canonically polarized varieties [**Kol13a**].

[1] In the language of [**BHJ17**, Section 7.2], the value equals $I^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) - J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$.

The proof of Theorem 1.1 requires significant machinery from the Minimal Model Program. The strategy is to show that if X is not lc, then (X, L) admits a test configuration with negative Futaki invariant. To construct the destabilizing test configuration, Odaka uses results from the Minimal Model Program for lc pairs [**OX12**].

2.1. A result from the MMP. The proof of Theorem 1.1 uses the following result.

THEOREM 2.5 ([**OX12**]). *Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier. If X is not lc, then there exists a closed subscheme $Z \subset X$ such that*

(1) the blowup $B_Z X$ is normal,
(2) the exceptional locus of $Y \to X$ is $\operatorname{Supp}(K_{Y/X})$, and
(3) all the coefficients of $K_{Y/X}$ are < -1.

Note that condition (3) is the hardest to arrange. The proof of Theorem 2.5 follows from the existence of log canonical models.

REMARK 2.6 (Log canonical models). In [**OX12**], the authors show that if X is a normal variety with K_X is $\mathbb{Q}$-Cartier, then there exists a proper birational morphism $f : Y \to X$ such that

(i) $(Y, \Delta_Y := \operatorname{Exc}(f))$ is an lc pair and
(ii) $K_Y + \Delta$ is relatively ample over X.

The pair (Y, Δ_Y) is called the *log canonical model* of X and is unique up to isomorphism.

To construct $Z \subset X$ in Theorem 2.5 from the log canonical model, set

$$E := K_{Y/X} + \Delta.$$

Since E is exceptional and ample over X, the negativity lemma implies $-E$ is effective [**KM98**, Lemma 3.39]. Now, if we define Z as the subscheme cut out by

$$\mathcal{I} := f_* \mathcal{O}_Y(-mE),$$

where $m > 0$ is sufficiently divisible, then [**BHJ17**, Lemma 1.13] implies $B_Z X \simeq Y$ and $\mathcal{I} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-mE)$. Since $K_{Y/X} = -E - \Delta_Y$ and $E = \operatorname{Exc}(f)$, the coefficients of $K_{Y/X}$ are all < -1.

EXAMPLE 2.7. Let $h \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d with $n \geq 3$ such that $H = \{h = 0\} \subset \mathbb{A}^{n+1}$ has an isolated singularity at 0. As discussed in Example 1.5.3, there exists a log resolution $\widetilde{H} \to H$ given by blowing up the point $0 \in H$, and

$$K_{\widetilde{H}/H} = (n - d)F,$$

where F is the unique exceptional divisor.

By the above discussion, H is not lc precisely when $d > n + 1$. In this case, the log canonical model of H is $\widetilde{H}$ and $Z = \{0\}$ satisfies the conclusion of Theorem 2.5. In general, the choice of Z will be substantially more complicated.

2.2. Proof of Theorem 2.1. Let (X, L) be a polarized variety such that X is normal and K_X is $\mathbb{Q}$-Cartier. Assume X is not lc. We aim to show there is a normal test configuration $(\mathcal{X}, \mathcal{L})$ with $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) < 0$.

Let $Z \subset X$ be a closed subscheme satisfying the conclusion of Theorem 2.5 and set

$$\mathcal{I} = \mathcal{I}_{Z\times\mathbb{A}^1} + t^N \mathcal{O}_{X\times\mathbb{A}^1} \subset \mathcal{O}_{X\times\mathbb{A}^1},$$

where N is a positive integer. Consider the normalization of the blowup of $X \times \mathbb{A}^1$ along $\mathcal{I}$:

$$\mathcal{X} := \widetilde{B_{\mathcal{I}} X \times \mathbb{A}^1} \xrightarrow{g} X \times \mathbb{A}^1.$$

We may write $\mathcal{I} \cdot \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(-E)$ for some Cartier divisor E. Set

$$\mathcal{L}_\varepsilon := g^* L_{\mathbb{A}^1} - \varepsilon E,$$

which is ample over $\mathbb{A}^1$ when $0 < \varepsilon \ll 1$, since $L_{\mathbb{A}^1}$ is ample and $-E$ is relatively ample over $X \times \mathbb{A}^1$. Similar to Example 1.7, $(\mathcal{X}, \mathcal{L}_\varepsilon)$ is a test configuration of (X, L).[2] Additionally, since the coefficients of $K_{Y/X}$ are < -1, [**Oda13**, page 14] shows $\mathrm{Supp}(K_{\mathcal{X}/X\times\mathbb{A}^1}) = \mathrm{Exc}(f)$ and the coefficients of $K_{\mathcal{X}/X\times\mathbb{A}^1}$ are < 0, assuming N is sufficiently large and divisible.

Proof of Theorem 2.1. We aim to show $\mathrm{Fut}(\mathcal{X}, \mathcal{L}_\varepsilon) < 0$ when $0 < \varepsilon \ll 1$. To proceed, we use that modified intersection formula in Proposition 1.5 implies

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}_\varepsilon) = \frac{\overline{\mathcal{L}}_\varepsilon^n \cdot K_{\overline{\mathcal{X}}/X\times\mathbb{P}^1}}{V} + \frac{\overline{\mathcal{L}}_\varepsilon^n g^* p_1^* K_X}{V} + \frac{\overline{S}}{n+1}\frac{\overline{\mathcal{L}}_\varepsilon^{n+1}}{V}. \tag{2.1}$$

By the multlinearity of the intersection product, the equation shows $\mathrm{Fut}(\mathcal{X}, \mathcal{L}_\epsilon)$ is a polynomial function for $0 < \varepsilon \ll 1$. We will now analyze the lowest order coefficient of the polynomial.

We begin with the first term in (2.1). Write $K_{\overline{\mathcal{X}}/X\times\mathbb{P}^1} = \sum_F a_F F$, where the sum runs through prime divisors $F \subset \mathrm{Supp}(E)$, and set

$$d := \min_F \mathrm{codim}_{X\times 0} g(F).$$

We compute

$$\begin{aligned}
\overline{\mathcal{L}}_\varepsilon^n \cdot K_{\overline{\mathcal{X}}/X\times\mathbb{P}^1} &= \sum_{j=0}^n \epsilon^{n-j} \binom{n}{j} (\overline{g}^* L_{\mathbb{P}^1})^j \cdot (-E)^{n-j} \cdot K_{\overline{\mathcal{X}}/X\times\mathbb{P}^1} \\
&= \sum_F \sum_{j=0}^n \epsilon^{n-j} a_F \binom{n}{j} (\overline{g}^* L_{\mathbb{P}^1})^j \cdot (-E)^{n-j} \cdot F.
\end{aligned}$$

By Lemma 2.8 proven below,

$$\overline{\mathcal{L}}_\varepsilon^n \cdot K_{\overline{\mathcal{X}}/X\times\mathbb{P}^1} = c\varepsilon^d + \text{ higher order terms}$$

[2] When $N = 1$, $\mathcal{I} = \mathcal{I}_{Z\times 0}$ and $B_{\mathcal{I}} X \times \mathbb{A}^1$ is the deformation to the normal cone from Example 1.7.

for some rational number $c < 0$. Repeating a similar argument for the remaining two terms in the formula for $\mathrm{Fut}(\mathcal{X}, \mathcal{L}_\varepsilon)$ gives

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}_\varepsilon) = c\varepsilon^d + \text{ higher order terms.}$$

Thus, $\mathrm{Fut}(\mathcal{X}, \mathcal{L}_\varepsilon) < 0$ for $0 < \varepsilon \ll 1$ and, hence, (X, L) is not K-semistable. □

Lemma 2.8. *Keep the notation from the proof of Theorem 2.1. Additionally, let $M_1, \dots, M_s$ be line bundles on X, F a prime exceptional divisor on $\overline{\mathcal{X}}$, and set*

$$C := g^* M_{1,\mathbb{P}^1} \cdot \ldots \cdot g^* M_{s,\mathbb{P}^1} \cdot (-E)^{n-1} \cdot F.$$

(1) If $\dim g(F) < s$, then $C = 0$.
(2) If $\dim g(F) = s$ and $M_1, \dots, M_s$ are ample, then $C > 0$.

Proof. It suffices to prove the result when $M_1, \dots, M_s$ are ample. Indeed, for (1), this follows from the fact that $M_i \simeq A_i \otimes B_i^*$ for some very ample line bundles A_i and B_i on X. For (2), the reduction follows from replacing each M_i by a power that is very ample.

Now, choose general elements $H_i \in |M_i|$. If (1) holds, then

$$H_1 \cap \cdots \cap H_s \cap g(F) = \emptyset.$$

Hence,

$$g^* H_{1,\mathbb{P}^1} \cap \cdots \cap g^* H_{s,\mathbb{P}^1} \cap F = \emptyset,$$

which implies $C = 0$. If (2) holds, then

$$H_1 \cap \cdots \cap H_s \cap g(F)$$

is a union of general points of $g(F)$. Thus,

$$C = \mathcal{O}_Y(-E)|_W^{n-s},$$

where $W := g^* H_{1,\mathbb{P}^1} \cap \cdots \cap g^* H_{s,\mathbb{P}^1} \cap F$ is dimension $n - s$. Since $-E$ is relatively ample over $X \times \mathbb{P}^1$, $\mathcal{O}_Y(-E)|_W$ is ample, which implies $C > 0$. □

CHAPTER 4

Valuations and test configurations

In this section, we discuss the language of valuations and their connection to test configurations. This connection was first explored in depth by Boucksom, Hisamotto, and Jonsson [**BHJ17**]. The presentation and results in Section 2 follows *loc. cit.* closely.

Conventions: All schemes are defined over an algebraically closed field k of characteristic 0.

1. Valuations

Throughout this section, let X be a normal variety and $K := K(X)$ denote its function field.

Definition 1.1. A *valuation* v of K is a map $v : K^\times \to \mathbb{R}$ satisfying
(1) $v(fg) = v(f) + v(g)$
(2) $v(f + g) \geq \min\{v(f), v(g)\}$
(3) $v|_{k^\times} = 0$

By convention, we set $v(0) = +\infty$.

In the literature, such a map is often referred to as a real valuation that is trivial on the base field. Since we only consider such objects, we use the above terminology.

Example 1.2. The following are examples of valuations.
(1) Let $x \in X$ be a smooth point on a variety of dimension n. The order of vanishing at x is the valuation defined by
$$\mathrm{ord}_x(f) := \max\{d \,|\, f \in \mathfrak{m}_x\}$$
for $0 \neq f \in \mathcal{O}_{X,x}$. Since ord_x satisfies (1), (2), and (3) for $f, g \in \mathcal{O}_{X,x} \setminus 0$ it extends uniquely to a valuation $K^\times \to \mathbb{R}$ by setting
$$\mathrm{ord}_x(f/g) := \mathrm{ord}_x(f) - \mathrm{ord}_x(g).$$
We will often specify a valuation by stating its values on a local ring of X.
(2) Let $X = \mathbb{A}^2_{x,y}$ and consider the valuation v of $K(x, y)$ defined by sending $f = \sum_{a,b\in\mathbb{N}} c_{a,b} x^a y^b$, where $c_{a,b} \in k$, to
$$v(f) = \min\{a + b\sqrt{2} \,|\, c_{a,b} \neq 0\}.$$
This is an example of a monomial valuation. The values $v(x) = 1$ and $v(y) = \sqrt{2}$ are its weights.

(3) Let $X = \mathbb{A}^2_{x,y}$ and consider the ring morphism

$$\phi : k[x,y] \hookrightarrow k[[t]],$$

which sends $x \mapsto t$ and $y \mapsto e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, which is a well defined power series. Let w denote the composition

$$k(x,y)^\times \hookrightarrow k((t))^\times \xrightarrow{\mathrm{ord}_t} \mathbb{Z},$$

where the first map is induced by ϕ and ord_t is the t-adic valuation. For example,

$$v\big(y - x - \tfrac{x^2}{2!}\big) = \mathrm{ord}_t\big(\tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \cdots\big) = 3.$$

1.1. Divisorial valuations. A *divisor E over X* is the data of a proper birational morphism $\mu : Y \to X$ and a prime divisor $E \subset Y$ with Y normal. Since the local ring $\mathcal{O}_{Y,E}$ is a DVR, there is an induced valuation

$$\mathrm{ord}_E : K^\times \to \mathbb{Z},$$

which sends $0 \neq f \in K$ to the order of vanishing of $\mu^* f$ along E. A valuation of the form $c\,\mathrm{ord}_E$, where $c \in \mathbb{R}_{>0}$, is called a divisorial valuation.

The valuation ord_x in Example 1.2.1 is a divisorial valuation. Indeed, $\mathrm{ord}_x = \mathrm{ord}_F$, where F is the exceptional divisor of the blowup of X at x.

1.2. Quasi-monomial valuations. Let $Y \to X$ be a proper biraitonal morphism, $\eta \in Y$ be a smooth point, and $y_1, \ldots, y_r \in \mathcal{O}_{Y,\eta}$ local coordinates. Given a weight vector $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r_{\geq 0}$, we define a valuation v_α of K as follows. By Cohen's structure theorem, there is an isomorphism

$$\widehat{\mathcal{O}_{Y,\eta}} = k(\eta)[[y_1, \ldots, y_r]].$$

Hence, we can write $0 \neq f \in \mathcal{O}_{Y,\eta}$ uniquely as $f = \sum_{\beta \in \mathbb{N}^r} c_\beta y^\beta$, where $c_\beta \in k(\eta)$ and $y^\beta := y_1^{\beta_1} \cdot \ldots, y_r^{\beta_r}$. We set

$$v_\alpha(f) := \min\{\langle \alpha, \beta \rangle \,|\, c_\beta \neq 0\}.$$

A valuation that can be written in this form is called *quasi-monomial*.

The case when $\eta \in Y$ is a codimension 1 point shows that all divisorial valuations are quasi-monomial. More generally, if $\alpha \in \mathbb{R} \cdot \mathbb{Z}^r_{\geq 0}$, then v_α is divisorial by Theorem 1.3.

1.3. Valuation rings. Let v be a valuation of K. The *valuation ring* of v is the local ring

$$\mathcal{O}_v := \{f \in K \,|\, v(f) \geq 0\},$$

which has maximal ideal $\mathfrak{m}_v := \{f \in K \mid v(f) \geq 0\}$. If there is a morphism $\operatorname{Spec}(\mathcal{O}_v) \to X$ such that the diagram

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}(\mathcal{O}_v) \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}(k) \end{tikzcd}$$

commutes, then the image of the closed point of $\operatorname{Spec}(\mathcal{O}_v)$ on X is called the *center* of v on X and denoted $c_X(v) \in X$. (Equivalently, the center of v is the point $\xi \in X$ such that $v \geq 0$ on $\mathcal{O}_{X,\xi}$ and $v > 0$ on $\mathfrak{m}_\xi$.) Since X is a variety, the valuative criterion for separatedness implies the center is unique if it exists. If X is proper, then the valuative criterion for properness implies that a center exists. We say v is a *valuation on* X if it has a center on X.

The *transcendence degree* of a valuatin v is

$$\operatorname{tr.deg}(v) := \operatorname{tr.deg}_k(k(v)),$$

where $k(v) := \mathcal{O}_v/\mathfrak{m}_v$. The *rational rank* of v is

$$\operatorname{rt.rk}(v) = \dim_{\mathbb{Q}}(\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q}),$$

where $\Gamma_v := v(K^\times)$ is the value group and $\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q}$ is viewed as a $\mathbb{Q}$-vector space. The Abhyankar inequality [**Abh56**] states that

$$\operatorname{tr.deg}(v) + \operatorname{rt.rk}(v) \leq n. \tag{1.1}$$

The case when equality holds is particularly important.

THEOREM 1.3. *Let v be a valuation of K. The following hold:*

(1) v is quasi-monomial if and only if $\operatorname{tr.deg}(v) + \operatorname{rt.rk}(v) = n$*;*
(2) v is divisorial if and only if $\operatorname{tr.deg}(v) = n - 1$ *and* $\operatorname{rt.rk}(v) = 1$.

Statement (1) is proven in [**JM12**, Proposition 3.7], which uses an argument of [**ELS03**]. Statement (2) is a classical theorem of Zariski [**KM98**, Lemma 2.45].

In the literature valuations such that the Abhyankar inequality is an equality are often referred to as Abhyankar valuations. The above theorem shows that in our setting, i.e. real valuations on the function field of characteristic 0 varieties, Abhyankar and quasi-monomial valuations are the same.

It is instructive to compute the transcendence degree and rational rank of the valuations in Example 1.2. These are given by

	tr.deg	rt.rk	dim(X)
ord_x	1	$n-1$	n
v	0	2	2
w	0	1	2

1.4. Valuations ideals. If v is a valuation of K, we can define its valuation ideal $\mathfrak{a}_\lambda(v) \subset \mathcal{O}_X$ for $\lambda \in \mathbb{R}_{\geq 0}$ as flows. If $U \subset X$ is an open set containing $c_X(v)$, then

$$\mathfrak{a}_\lambda(v) := \{f \in \mathcal{O}_X(U) \,|\, v(f) \geq \lambda\}.$$

If $c_X(v) \notin U$, then $\mathfrak{a}_\lambda(v) = \mathcal{O}_X(U)$.

Remark 1.4. There are also valuation ideals on $\mathcal{O}_v$ defined by $I_\lambda(v) := \{f \in \mathcal{O}_v \,|\, v(f) \geq \lambda\} \subset \mathcal{O}_v$. The two types of valuation ideals are related by $\mathfrak{a}_\lambda(U) = \mathcal{O}_X(U) \cap I_\lambda(v)$ when $c_X(v) \in U$

1.5. Evaluation. Using the notion of the center, a valuation can evaluate more general objects than functions.

Definition 1.5. Fix a valuation $v \in \mathrm{Val}_X$.

(1) For an ideal $\mathfrak{a} \subset \mathcal{O}_X$, we set $v(\mathfrak{a}) := \min\{f \in \mathfrak{a} \cdot \mathcal{O}_{X,c_X(v)}\}$.
(2) For a Cartier divisor D on X, $v(D) := v(f_D)$, where f_D is the local defining equation of D at $c_X(v)$.
(3) More generally, for a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D, we set $v(D) := \frac{1}{m}v(mD)$, where m is a positive integer such that mD is a Cartier divisor.
(4) Let L be a line bundle and $s \in H^0(X, mL)$. After fixing a trivialization $\phi : L_{c_X(v)} \xrightarrow{\simeq} \mathcal{O}_{X,c_X(v)}$ at $c_X(v)$, we set $v(s) := v(\phi(s))$.

In (2)-(4), the value is independent of choice. For example, for (2) this uses that any other local defining equation differs from f_D by a unit in $\mathcal{O}_{X,c_X(v)}$.

1.6. Valuation space. We write Val_X for the set of valuation on X. The set Val_X has the structure of a topological space, where the topology is the weakest topology such that the function

$$\mathrm{ev}_f : \mathrm{Val}_X \to \mathbb{R},$$

defined by $\mathrm{ev}_f(v) := v(f)$, is continuous for each $f \in K^\times$.

Remark 1.6 (Berkovich spaces). A fundamental class of objects in non-Archimedean geometry are Berkovich spaces. The space Val_X is a dense subset of the Berkovich space of X with respect to the trivial valuation on k. The boundary parameterizes valuations on subvarieties of X. See [**JM12**, Section 6.3] for further discussion on the relation.

1.7. Log discrepany. Let X be a normal variety such K_X is a $\mathbb{Q}$-Cartier.

Definition 1.7. If E is a prime divisor on a normal variety Y with a proper birational morphism $Y \to X$, then the *log discrepancy* of E

$$A_X(E) = 1 + \mathrm{coeff}_E(K_{Y/X}).$$

We will often write $A_X(\mathrm{ord}_E)$ for this value.

Remark 1.8 (Relation to lc and klt). Using that $K_{Y/X} = \sum_{E \subset Y}(A_X(E) - 1)E$, it follows that:

(1) X is klt if and only if $A_X(E) > 0$ for all divisors E over X;
(2) X is lc if and only if $A_X(E) \geq 0$ for all divisors E over X.

REMARK 1.9 (Log discrepancy vs. discrepancy). In the birational geometry literatuere one often considers the *discrepancy* of E [**KM98**, Section 2.3], which is

$$a(E; X) := \mathrm{coeff}_E(K_{Y/X}) = A_X(E) - 1.$$

While both notions can be used to define klt (resp., lc) singularities, the log discrepancy extends more naturally to a continuous function on the valuation space. This can be seen in the following example.

EXAMPLE 1.10. Let $X = \mathbb{A}^2_{x,y}$. For relativlye prime positive integers a, b, let $Y_{a,b} \to \mathbb{A}^2$ denote the weighted blowup at 0 with weights a and b along x and y with exceptional divisor $F_{a,b}$. The valuation $\mathrm{ord}_{F_{a,b}}$ is the monomial valuation on $\mathbb{A}^2$ with weights a and b along x and y.

A toric computation shows

$$A_{\mathbb{A}^2}(F_{a,b}) = a + b.$$

Hence, the log discrepancy is linear in a and b, while the discrepancy is not.

THEOREM 1.11. [**JM12**,**Bou15**] *The log discrepancy extends uniquely to a lower-semicontinuous, homogenous function*

$$\mathrm{Val}_X \to \mathbb{R} \cup \{\infty\}.$$

The function can be defined in the following steps.

(1) For a divisorial valuation $c\mathrm{ord}_E$,
$$A_X(c\mathrm{ord}_E) := cA_X(E) = c(1 + \mathrm{coeff}_E(K_{Y/X})$$
(2) For a quasi-monomial valuation v_α in the notation of Section 1.2, we extend linearly by setting
$$A_X(v_\alpha) = \alpha_1 A_X(E_1) + \cdots \alpha_r A_X(E_r),$$
where E_i is the prime divisor on Y locally defined by y_i at η,
(3) For a general valuation v, $A_X(v)$ is defined by approximating v by quasi-monomial valuations and taking a sup. See [**JM12**, Section 5.2] for details.

2. Relations to test configurations

In this section, we describe a link between K-stability and valuations observed in [**BHJ17**]. The connection arises from the fact that a normal test configuration $(\mathcal{X}, \mathcal{L})$ of a polarized pair (X, L) induces a canonical $\mathbb{G}_m$-equivariant birational map

$$\mathcal{X} \dashrightarrow X \times \mathbb{A}^1$$

and each irreducible component $E \subset \mathcal{X}_0$ induces a valuation ord_E of $K(\mathcal{X})$, which is isomorphic to $K(X)(t)$.

2.0.1. *Valuations on the trivial test configuration.* Let X be a normal variety. In this section, we study valuations $K(X \times \mathbb{A}^1) \simeq K(X)(t)$.

DEFINITION 2.1. If v is a valuation of $K(X)(t)$, we denote by $r(v)$ its restriction to $K(X)$, which fits into a diagram

$$\begin{tikzcd} K(X)^\times \arrow[r, hook] \arrow[rr, bend left, "r(v)"] & K(X)(t)^\times \arrow[r, "v"] & \mathbb{R} \end{tikzcd} .$$

PROPOSITION 2.2. *Let v be a valuation of $K(X)(t)$.*

(1) If v is quasi-monomial, then $r(v)$ is quasi-monomial.
(2) If v is divisorial, then $r(v)$ is divisorial or the trivial valuation.

PROOF. The proof uses a generalized version of the Abhyankar inequality, which states that if $k \subset L' \subset L$ are field extensions and w is a valuation of L, then

$$\text{tr.deg} w + \text{rt.rk} w \leq \text{tr.deg} w' + \text{rt.rk} w' + \text{tr.deg}_{L'}(L),$$

where w' is its restriction L' [**BHJ17**, (1.3)]. Therefore,

$$\text{tr.deg} v + \text{rt.rk} v \leq \text{tr.deg} r(v) + \text{rt.rk} r(v) + 1 \leq n + 1,$$

where the second inequality is (1.1). Since v is quasi-monomial, we know

$$\text{tr.deg} v + \text{rt.rk} v = n + 1$$

by Theorem 1.3. Combining the previous two inequalities gives

$$\text{tr.deg} r(v) + \text{rt.rk} r(v) = n.$$

Thus, $r(v)$ is quasi-monomial by Theorem 1.3. If v is in addition divisorial, then

$$\text{rt.rk} r(v) \leq \text{rt.rk} v = 1.$$

If $\text{rt.rk} r(v) = 1$, then $r(v)$ is divisorial by Theorem 1.3. If $\text{rt.rk} r(v) = 0$, then $r(v)$ is the trivial valuation. □

DEFINITION 2.3. A valuation v of $K(X)(t)$ is $\mathbb{G}_m$-*equivariant* if

$$v(f) = v(f \cdot a)$$

for each $f \in K(X)(t)$ and $a \in \mathbb{G}_m(k) = k^\times$. We write $\text{Val}^{\mathbb{G}_m}_{X \times \mathbb{A}^1}$ for the set of $\mathbb{G}_m$-equivariant valuations on $X \times \mathbb{A}^1$.

In the above definition, the action of $a \in k^\times$ on $f \in K(X)(t)$ is given by

$$a \cdot f = \sum_{\lambda \in \mathbb{Z}} a^{-\lambda} f_\lambda t^\lambda,$$

where $f = \sum_{\lambda \in \mathbb{Z}} f_\lambda t^\lambda$ and each $f_\lambda \in K(X)$.

EXAMPLE 2.4. Fix a valuation w of $K(X)$ and a real number $s \geq 0$. We define a valuation w_s of $K(X)(t)$ by setting

$$w_s(f) := \min\{w(f_\lambda) + \lambda s\}$$

for $f = \sum_\lambda f_\lambda t^\lambda \in K(X)(t)$. We state a number of properties of w_s.

(1) The valuation is $\mathbb{G}_m$-equivariant, since $w(a^{-\lambda} f_\lambda) = w(f_\lambda)$ for each $a \in k^\times$.
(2) If w has center on X, then

$$c_{X \times \mathbb{A}^1}(w_s) = \begin{cases} c_X(w) \times 0 & \text{if } s > 0 \\ c_X(w) \times \mathbb{A}^1 & \text{if } s = 0 \end{cases}$$

(3) If $w = c \cdot \mathrm{ord}_F$ for some integer c and prime F on a variety Y with a proper birational morphism $Y \to X$, then w_s is the quasi-monomial combination of $F \times \mathbb{A}^1$ and $Y \times 0$ with weights c and s.

PROPOSITION 2.5. *The map*

$$\mathrm{Val}_X \times \mathbb{R} \to \mathrm{Val}^{\mathbb{G}_m}_{X \times \mathbb{A}^1},$$

sending (w, s) *to* w_s *is a bijection.*

PROOF. The map is clearly injective. Hence, it suffices to show surjectivity. Fix a $\mathbb{G}_m$ equivariant valuation on $X \times \mathbb{A}^1$. Set

$$w := r(v) \quad \text{and} \quad s := v(t).$$

We seek to show $v = w_s$.

To proceed, fix $f = \sum_\lambda f_\lambda t^\lambda \in K(X)(t)$ and observe that

$$v(f) \geq \min\{v(f_\lambda) + \lambda s) \,|\, \lambda \in \mathbb{Z}\} = w_s(f).$$

To prove the reverse inequality, fix a positive integer N such that $f_\lambda = 0$ for $\lambda > N$. Next, fix $\lambda_0 \leq N$ achieving the minimum, i.e.

$$v(f_{\lambda_0}) + \lambda s = \min\{v(f_\lambda) + \lambda s) \,|\, \lambda \in \mathbb{Z}\},$$

and distinct elements $(a_\mu)_{1 \leq \mu \leq N}$ in $k^\times$. Using that

$$a_\mu \cdot f = \sum_{\lambda \in \Lambda} a_\mu^{-\lambda} f_\lambda t^\lambda$$

and that the determinant of $(a_\mu^{-\lambda})_{1 \leq \mu, \lambda \leq N}$ is non-zero by the Vandermonde determinant formula, we see $f_{\lambda_0} t^{\lambda_0}$ is a k-linear combination of the $a_\mu \cdot f$. Hence,

$$w_s(f) = v(f_{\lambda_0} t^{\lambda_0}) \geq \min\{v(a_\mu \cdot f) \,|\, 1 \leq \mu \leq N\} = v(f),$$

which implies $w_s(f) = v(f)$ as desired. Therefore, $w_s = v$. $\square$

2.1. Test configurations. Let $(\mathcal{X}, \mathcal{L})$ be a normal test configuration of polarized variety (X, L). Consider the diagram

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathcal{X} \arrow[rr, dashed] & & X \times \mathbb{A}^1 \end{tikzcd},$$

where $\mathcal{Y}$ is the main component of the normalization of the graph of the canonical birational map $\mathcal{X} \dashrightarrow X \times \mathbb{A}^1$. Note that the above morphisms are $\mathbb{G}_m$-equivariant, where the action on $X \times \mathbb{A}^1$ is the product of the trivial action on X and the standard action on $\mathbb{A}^1$.

DEFINITION 2.6. An irreducible component $E \subset \mathcal{X}_0$ induces a valuation ord_E of $K(\mathcal{X}) \simeq K(X)(t)$. We write $v_E := r(\mathrm{ord}_E)$ for its restriction to $K(X)$.

We state a number of important properties of these valuations

LEMMA 2.7. *Let E be an irreducible component of $\mathcal{X}_0$.*

(1) ord_E *is a $\mathbb{G}_m$-equivariant divisorial valuation on $X \times \mathbb{A}^1$.*
(2) $v_E = c\,\mathrm{ord}_F$ *for some integer $c > 0$ and prime divisor F on a normal variety Y with a proper birational morphism $Y \to X$.*

Additionally, ord_E *is the quasi-monomial valuation with weights c and* $\mathrm{ord}_E(t)$ *along $F \times \mathbb{A}^1$ and $Y \times 0$.*

PROOF. We first prove (1). Since $\mathbb{G}_m$ and E are irreducible,

$$\mathbb{G}_m \cdot E = \mathrm{im}(\mathbb{G}_m \times E \to \mathcal{X})$$

is irreducible. Using that $\mathbb{G}_m \cdot E \subset \mathcal{X}_0$ and $1 \cdot E = E$, it follows that $\mathbb{G}_m \cdot E = E$. Therefore, $a \cdot E$ for each $a \in \mathbb{G}_m(k)$, and, hence, ord_E is $\mathbb{G}_m$-equivariant. Since $\mathrm{ord}_E = \mathrm{ord}_{\widetilde{E}}$, where $\widetilde{E}$ is the birational transform of E on $\mathcal{Y}$, ord_E is a divisorial valuation on $X \times \mathbb{A}^1$.

Statement (2) follows immediately from Theorem 1.3.2 and that $\Gamma_{v_E} \subset \mathbb{Z}$. The last statement is a consequence of Example 2.4 and Proposition 2.5. □

Using the valuations in Definition 2.6, we can describe the filtration of the section ring of (X, L) induced by the test configuration.

PROPOSITION 2.8. *For $m > 0$ sufficiently large and divisible,*

$$F^{\lambda}_{\mathcal{X},\mathcal{L}} H^0(X, mL) := \bigcap_{E \subset \mathcal{X}_0} \left\{ s \in H^0(X, mL) \,|\, v_E(s) + m\mathrm{ord}_E(D) \geq \lambda \mathrm{ord}_E(t) \right\},$$

where D denotes the $\mathbb{Q}$-divisor on $\mathcal{Y}$ supported on $\mathcal{Y}_0$ such that $f^*\mathcal{L} \simeq g^* L_{\mathbb{A}^1} + D$.

PROOF. Fix $s \in H^0(X, mL)$, and let $\overline{s}$ denote the $\mathbb{G}_m$-invariant section of $L \times (\mathbb{A}^1 \setminus 0)$ such that $\overline{s}_1 = s$. Using that $(\mathcal{X}, \mathcal{L})|_{\mathbb{A}^1 \setminus 0} \simeq (X, L) \times (\mathbb{A}^1 \setminus 0)$, we may view $\overline{s}$ as a rational section of $m\mathcal{L}$.

By Remark 1.14, $s \in F^\lambda H^0(X, mL)$ if and only if $\overline{s}t^{-\lambda} \in H^0(\mathcal{X}, \mathcal{L})$. Since $\mathcal{X}$ is normal, $\overline{s}t^{-\lambda} \in H^0(\mathcal{X}, \mathcal{L})$ if and only if $\mathrm{ord}_E(\overline{s}t^{-\lambda}) \geq 0$ for all irreducible components $E \subset \mathcal{X}_0$. We compute

$$\mathrm{ord}_E(\overline{s}t^{-\lambda}) = \mathrm{ord}_E(\overline{s}) - \lambda \mathrm{ord}_E(t) = v_E(s) + m\mathrm{ord}_E(D) - \lambda \mathrm{ord}_E(t).$$

Hence, the desired formula holds. □

REMARK 2.9. The last equality in the proof is a tad confusing, since $\overline{s}$ gives a rational section of $m\mathcal{L}$ as well as $mL_{\mathbb{A}^1}$. The above choice changes the value of ord_E.

Adding subscripts to denote which line bundle the rational section lives on, we compute in more detail:

$$\begin{aligned}
\mathrm{ord}_E(\overline{s}t^{-\lambda}_{m\mathcal{L}}) = \mathrm{ord}_E(\overline{s}_{m\mathcal{L}}) - \lambda \mathrm{ord}_E(t) &= \mathrm{ord}_E(\overline{s}_{f^*m\mathcal{L}}) - \lambda \mathrm{ord}_E(t) \\
&= \mathrm{ord}_E(\overline{s}_{g^*mL_{\mathbb{A}^1}(D)}) - \lambda \mathrm{ord}_E(t) \\
&= \mathrm{ord}_E(\overline{s}_{g^*mL_{\mathbb{A}^1}}) + m\mathrm{ord}_E(D) - \lambda \mathrm{ord}_E(t) \\
&= v_E(s) + m\mathrm{ord}_E(D) - \lambda \mathrm{ord}_E(t).
\end{aligned}$$

The following comparison of log discrepances will be useful in future sections.

LEMMA 2.10. *If E is an irreducible component of $\mathcal{X}_0$, then*

$$A_{X\times\mathbb{A}^1}(\mathrm{ord}_E) = A_X(v_E) + \mathrm{ord}_E(t).$$

PROOF. Using the notation and statement of Lemma 2.7, we compute

$$\begin{aligned}
A_{X\times\mathbb{A}^1}(\mathrm{ord}_E) &= c \cdot A_{X\times\mathbb{A}^1}(\mathrm{ord}_{F\times\mathbb{A}^1}) + \mathrm{ord}_E(t) \cdot A_{X\times\mathbb{A}^1}(\mathrm{ord}_{Y\times 0}) \\
&= c \cdot A_X(\mathrm{ord}_F) + \mathrm{ord}_E(t) \\
&= A_X(v_{\mathcal{X}_0}) + \mathrm{ord}_E(t),
\end{aligned}$$

where the first equality uses that ord_E is the quasi-monomial combination of $F \times \mathbb{A}^1$ and $Y \times 0$ with weights c and $\mathrm{ord}_E(t)$, the second that $K_{Y\times\mathbb{A}^1/X\times\mathbb{A}^1} = p_1^* K_{Y/X}$, and the third that $v_E = c\mathrm{ord}_F$. □

CHAPTER 5

K-stability of Fano varieties and Valuations

In this chapter, we develop tools for understanding the K-stability of Fano varieties. In particular, we discuss Kento Fujita and Chi Li's valuative criterion for K-stability.

Conventions: Throught this section, all schemes are defined over an algebraically closed field k of characteristic 0.

1. Klt Fano varieties

1.1. Definition.

DEFINITION 1.1. A *klt Fano variety* is a projective variety X such that X is klt and $-K_X$ is ample.

REMARK 1.2 (Why klt?). Of course, the most important examples of klt Fano varieties are smooth Fano varieties. Regardless, we are more generally interested in klt Fano varieties for the following reasons.

(1) A possibly singular normal K-semistable Fano variety has at worst klt singularities by [**Oda13**]. Hence, this is the largest class of Fano varieties for which studying K-stability makes sense.
(2) Most algebraic arguments in the study of the K-stability of smooth Fano varieties also apply verbatim to klt Fano varieties.
(3) As we will see in later chapters, smooth K-polystable Fano varieties are parameterized by a quasi-projective moduli space. The moduli space can be compactified by parameterizing smoothable klt K-polystable Fano varieties at the boundary.

EXAMPLE 1.3. Let us give a few interesting examples of klt Fano varieties.

(1) Since smooth varieties are klt, we already listed a number examples of klt Fano varieties in Section 1.3.
(2) Weighted projective space. Let $a_0, \ldots, a_n$ be positive integers and set

$$\mathbb{P}(a_1, \ldots, a_n) = \operatorname{Proj} k[x_0, \ldots, x_n],$$

where $k[x_0, \ldots, x_n]$ is the graded ring with x_i weight a_i. Set theoretically,

$$\mathbb{P}(a_1, \ldots, a_n)(k) = (\mathbb{A}^{n+1}(k) \setminus \mathbf{0})/ \sim,$$

where the equivalence relation is given by $(x_0, \ldots, x_{n+1}) \sim (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n)$ for each $\lambda \in k^\times$. Using that weighted projective spaces have quotient singularities, which are klt, and a computation of the canonical divisor, it follows that they are klt Fano varieties.

(3) Let $f \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree $d < n+1$ such that $X = \{f = 0\} \subset \mathbb{P}^n$ is a hypersurface. Its projectivized cone is $Y = \{f = 0\} \subset \mathbb{P}^{n+1}$, where f is viewed as a polynomial in $k[x_0, \ldots, x_{n+1}$. Note that Y is a klt Fano variety, since it is klt by Example 1.14 and has $-K_Y = \mathcal{O}_Y(-n-2+d)$, which is ample.

In this chapter, we will discuss the K-stability of klt Fano varieties. To reduce notation, a *test configuration* $(\mathcal{X}, \mathcal{L})$ of a klt Fano variety X will always mean a test configuration of $(X, -K_X)$.

1.2. Special test configurations. In the study of the K-stability of Fano varieties, a special class of test configurations play an important

DEFINITION 1.4. A test configuration $(\mathcal{X}, \mathcal{L})$ of a klt Fano variety X is *special* if $\mathcal{X}_0$ is a klt Fano variety.

EXAMPLE 1.5. The test configuration in Example 2.2, which is the degeneration of $\mathbb{P}^2$ to the cone over a conic, is a special test configuration.

In Tian's original definition of K-stability, he only considered special test configurations [**Tia97**].[1] The following theorem of Li and Xu shows that his definition agrees with Donaldson's definition [**Don02**].

THEOREM 1.6. [**LX14**] *To test K-(semi/poly)stability of a klt Fano variety, it suffices to consider special test configurations.*

More precisely, [**LX14**] shows that if $(\mathcal{X}, \mathcal{L})$ is a test configuration of a klt Fano variety X, then there exists an integer $d > 0$ and special test configuration $(\mathcal{X}^s, -K_{\mathcal{X}^s/\mathbb{A}^1})$ of X such that

$$\mathrm{Fut}(\mathcal{X}^s, -K_{\mathcal{X}^s/\mathbb{A}^1}) \leq d\mathrm{Fut}(\mathcal{X}, \mathcal{L}).$$

Its proof relies on using the Minimal Model Program to modify the original test configuration and showing that the Futaki invariant decreases throughout this process.

REMARK 1.7 (Integral central fiber). If $(\mathcal{X}, \mathcal{L})$ is a special test configuration, then $\mathcal{X}_0$ is variety and, hence, integral. In the next sections, test configurations with $\mathcal{X}_0$ integral will play an important role due to their relation to certain valuations on X.

2. Test configurations and dreamy valuations

Throughout, let X be a klt Fano variety and fix $r > 0$ such that $L := -rK_X$ is a line bundle.

[1]The definition of special in [**Tia97**] is that $\mathcal{X}_0$ is normal. This differs slightly from our definition.

2.1. Dreamy valuations. A $\mathbb{Z}$-valued divisorial valuation v on X induces a $\mathbb{Z}$-filtration of $R(X,L) := \bigoplus_{m\in\mathbb{N}} H^0(X,mL)$ defined by

$$F_v^\lambda H^0(X,mL) := \{s \in H^0(X,mL) \,|\, v(s) \geq \lambda\}.$$

To see this is indeed a $\mathbb{Z}$-filtration, note that

$$F_v^\lambda H^0(X,mL) \cdot F_v^\mu H^0(X,qL) \subset F_v^{\lambda+\mu} H^0(X,(m+q)L),$$

by the definition of a valuation.

REMARK 2.1. Note that if we write $v = c \cdot \mathrm{ord}_E$ and $E \subset Y \xrightarrow{\mu} X$, then we have a diagram

$$\begin{tikzcd} F_v^\lambda H^0(X,mL) \arrow[r] \arrow[d, "\mu^*"] & H^0(X,mL) \arrow[d, "\mu^*"] \\ H^0\big(Y, m\mu^*L - \lfloor \lambda/c \rfloor E\big) \arrow[r, hook] & H^0(Y, m\mu^*L) \end{tikzcd},$$

where the vertical arrows are isomorphisms. It will be convenient to view the filtration using this isomorphism.

DEFINITION 2.2 (Fujita). A $\mathbb{Z}$-valued divisorial valuation v of X is *dreamy* if $F_v^\bullet$ is finite generated. If $v = \mathrm{ord}_E$ is dreamy, we say E is a *dreamy* divisor over X.

The work *dreamy* is a reference to *Mori dream spaces*, which were defined by Hu and Keel in [**HK00**]. Such spaces satisfy many finiteness properties.

REMARK 2.3. If we write $v = c \cdot \mathrm{ord}_E$ and $E \subset Y \xrightarrow{\mu} X$, then Remark 2.1 implies $\mathrm{Rees}(F_v^\bullet)$ is isomorphic to

$$\bigoplus_{(m,\lambda)\in\mathbb{N}\times\mathbb{Z}} H^0(Y, m\mu^*L - \lfloor \tfrac{\lambda}{c} \rfloor E)$$

Hence, v is dreamy if this algebra is finitely generated.

EXAMPLE 2.4. We give a few examples of dreamy and non-dreamy divisors.

(1) If E is a prime divisor on X (i.e. $E \subset X$), then E is dreamy.
   (a) In the case when $E \sim_{\mathbb{Q}} -cK_X$ for some $c \in \mathbb{Q}$ (e.g. $X = \mathbb{P}^n$ and $E \subset \mathbb{P}^n$ is any prime divisor), this can be deduced from the fact that $R(X,L)$ is finitely generated.
   (b) The general result is highly non-trivial and follows from a powerful finite generation result in [**BCHM10**, Corollary 1.1.9] and the assumption that X is Fano.

(2) As we will see in Proposition 2.6, if $(\mathcal{X},\mathcal{L})$ is a test configuration of X with $\mathcal{X}_0$ integral, then the valuation $v_{\mathcal{X}_0}$ on X is dreamy.

(3) As observed by Goto, Nishida and Watanabe, there exist space $C \subset \mathbb{P}^3$ such that the

$$\bigoplus_{\lambda\in\mathbb{N}} \mathfrak{a}_\lambda(\mathrm{ord}_C)$$

is not finitely generated, where $\mathfrak{a}_\lambda(\mathrm{ord}_C) \subset \mathcal{O}_{\mathbb{P}^2}$ is the λ-th valuation ideal [**Laz04**, Remark 2.4.17]. From this, it can be deduced that ord_C is not dreamy.

In general, one expects that "most" valuations are not dreamy.

2.2. Test configurations with integral central fiber. As we will see, dreamy valuations are related to test configurations with integral central fiber. To begin, we observe some basic properties of such test configurations.

LEMMA 2.5. *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration of a klt Fano variety X. If $\mathcal{X}_0$ is integral, then*

(1) $\mathcal{X}$ is normal and
(2) $\mathcal{L} \simeq -K_{\mathcal{X}/\mathbb{A}^1} + c\mathcal{X}_0$ for some rational number $c \in \mathbb{Q}$.

PROOF. Since X is normal by the klt assumption and $\mathcal{X}_0$ is reduced, Proposition 1.21.3 implies $\mathcal{X}$ is normal. Next, fix an integer r such that $r\mathcal{L}$ is line bundle. Since

$$r\mathcal{L}|_{\mathbb{A}^1\setminus 0} \simeq -rp^*K_X \simeq -rK_{\mathcal{X}/\mathbb{A}^1}|_{\mathbb{A}^1\setminus 0},$$

where p is the composition $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{A}^1 \setminus 0) \to X$, there exists a divisor G supported on $\mathcal{X}_0$ such that

$$r\mathcal{L} \simeq -rK_{\mathcal{X}/\mathbb{A}^1} + G.$$

Since $\mathcal{X}_0$ is a prime Cartier divisor, G is multiple of $\mathcal{X}_0$, which implies (2) holds. □

Since a scheme is integral if and only if it is irreducible and reduced, Lemma 2.5.1 implies $\mathrm{ord}_{\mathcal{X}_0}$ is a divisorial valuation of $\mathcal{X}$.

PROPOSITION 2.6. *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration of a klt Fano variety X with $\mathcal{X}_0$ integral. The following hold:*

(1) There exists a rational number C such that

$$F^{\lambda}_{\mathcal{X},\mathcal{L}} H^0(X, L) = F^{\lambda+mC}_{v_{\mathcal{X}_0}} H^0(X, mL)$$

for all $m \in \mathbb{N}$ and $\lambda \in \mathbb{Z}$.
(2) If $\mathcal{L} = -K_{\mathcal{X}/\mathbb{A}^1}$ as linearized $\mathbb{Q}$-line bundles, then $C = -rA_X(v_0)$.
(3) The valuation $v_{\mathcal{X}_0}$ is dreamy.

PROOF. By Proposition 2.8, we have

$$F^\lambda H^0(X, mL) = \left\{ s \in H^0(X, mL) \,|\, v_{\mathcal{X}_0}(s) + mr\mathrm{ord}_{\mathcal{X}_0}(D) \geq \lambda \mathrm{ord}_{\mathcal{X}_0}(t) \right\}$$

for $m > 0$ sufficiently divisible. The $\mathbb{Q}$-divisor D appearing above is defined as follows. Consider the diagram

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathcal{X} \arrow[rr, dashed] & & X \times \mathbb{A}^1, \end{tikzcd}$$

where $\mathcal{Y}$ is the main component of the normalization of the graph of $\mathcal{X} \dashrightarrow X \times \mathbb{A}^1$. Above, D is the $\mathbb{Q}$-divisor supported on $\mathcal{X}_0$ such that $f^*\mathcal{L} = g^* p_1^*(-K_X)(D)$. Since $\mathcal{X}_0 = \{t = 0\}$ and $\mathcal{X}_0$ is reduced, $\mathrm{ord}_{\mathcal{X}_0}(t) = 1$. Thus, the first statement holds with $C := \mathrm{ord}_{\mathcal{X}_0}(D)$.

By (1), there is an isomorphism of $k[t]$-algebras

$$\mathrm{Rees}(F_v^\bullet) \simeq \mathrm{Rees}(F_{\mathcal{X},\mathcal{L}}^\bullet).$$

(The isomorphism is not an isomorphism of $\mathbb{N} \times \mathbb{Z}$-graded algebras, since there is a shift in grading). When $\mathcal{L} = -K_{\mathcal{X}/\mathbb{A}^1}$, we compute

$$\begin{aligned} D &= g^*(K_{X\times\mathbb{A}^1} - p_2^* K_{\mathbb{A}^1}) - f^*(K_{\mathcal{X}} - \pi^* K_{\mathbb{A}^1}) \\ &= g^* K_{X\times\mathbb{A}^1} - f^* K_{\mathcal{X}} \\ &= K_{\mathcal{Y}/\mathcal{X}} - K_{\mathcal{Y}/X\times\mathbb{A}^1}, \end{aligned}$$

where the first equality uses that $K_{X\times\mathbb{A}^1} = p_1^* K_X + p_2^* K_{\mathbb{A}^1}$ and the second that $g \circ p_2 = f \circ \pi$. Therefore,

$$\begin{aligned} \mathrm{ord}_{\mathcal{X}_0}(D) = \mathrm{coeff}_{\widetilde{\mathcal{X}_0}}(K_{\mathcal{Y}/\mathcal{X}}) - \mathrm{coeff}_{\widetilde{\mathcal{X}}_0} K_{\mathcal{Y}/X\times\mathbb{A}^1} &= 0 - (A_{X\times\mathbb{A}^1}(\mathrm{ord}_{\mathcal{X}_0}) - 1) \\ &= -A_X(v_{\mathcal{X}_0}), \end{aligned}$$

where the last equality is Lemma 2.10. $\square$

2.3. Correspondence.

Proposition 2.7. *There exists a bijective correspondence between*

(1) test configurations $(\mathcal{X}, -K_{\mathcal{X}/\mathbb{A}^1})$ of X with $\mathcal{X}_0$ integral
(2) dreamy $\mathbb{Z}$-valued divisorial valuations of X.

given by the map sending a test configuration $(\mathcal{X}, -K_{\mathcal{X}/\mathbb{A}^1})$ to the valuation $v_{\mathcal{X}_0}$.

Example 2.8. We provide a few simple examples of this correspondence:

(1) The map sends the trivial test configuration

$$(X \times \mathbb{A}^1, -K_{X\times\mathbb{A}^1/\mathbb{A}^1})$$

to the trivial valuation v_{triv} of X.

(2) Consider a product test configuration

$$\mathcal{X} := X \times \mathbb{A}^1$$

induced by a $\mathbb{G}_m$-action $\sigma : \mathbb{G}_m \times X \to X$ As observed in [?Fuj19, Example 3.5], the induced valuation $v_{\mathcal{X}_0}$ is the composition

$$\begin{tikzcd} K(X)^\times \arrow[r, "\sigma^*"'] \arrow[rr, bend left, "v_{\mathcal{X}_0}"] & K(X)(t)^\times \arrow[r, "\mathrm{ord}_{t-1}"'] & \mathbb{Z} \end{tikzcd} ,$$

where $\sigma : \mathbb{G}_m \times X \to X$ is the action of $\mathbb{G}_m$ on X.[2] We call $v_{\mathcal{X}_0}$ a *product valuations*, since it is induced by product test configurations.

(2') The choice of integers a and b, induces a $\mathbb{G}_m$-action on $\mathbb{P}^2_{[x_0:x_1:x_2]}$ given by

$$t \cdot [x_0 : x_1 : x_2] = [x_0, t^{-a}x_1 : t^{-b}x_2]$$

and, hence, a product test configuration $\mathcal{X}^{a,b} := \mathbb{P}^2 \times \mathbb{A}^1$ of $\mathbb{P}^2$. Using (2), we see the induced valuation is the monomial valuation on

$$\mathbb{A}^2 \simeq U_{x_0 \neq 0} \hookrightarrow \mathbb{P}^2,$$

with weights a and b on $\frac{x_1}{x_0}$ and $\frac{x_2}{x_0}$.

To prove the correspondence, we need the following lemma, which describes the test configuration induced by a dreamy valuation.

Lemma 2.9. *Let v be dreamy $\mathbb{Z}$-valued divisorial valuations of X and set*

$$\mathcal{X} := \mathrm{Proj}_{\mathbb{A}^1}\big(\mathrm{Rees}(F_v^\bullet)\big),$$

The following hold:

(1) $(\mathcal{X}, -K_{\mathcal{X}/\mathbb{A}^1})$ is a normal test configuration of X
(2) $\mathcal{X}_0$ is integral, and
(3) $v_{\mathcal{X}_0} = v$.

Note that it only makes sense to consider $\mathcal{X} := \mathrm{Proj}_{\mathbb{A}^1}\big(\mathrm{Rees}(F_v^\bullet)\big)$ in the case when v is dreamy, since otherwise $\mathrm{Rees}(F_v^\bullet)$ is not finitely generated and its Proj will not necessarily be projective over $\mathbb{A}^1$.

Proof. To prove $\mathcal{X}_0$ is integral, note that

$$\frac{\mathrm{Rees}(F_v^\bullet)}{t\mathrm{Rees}(F_v^\bullet))} = \bigoplus_{m\in\mathbb{N}} \bigoplus_{\lambda\in\mathbb{Z}} \frac{F_v^\lambda H^0(X, mL)}{F_v^{\lambda+1} H^0(X, mL)} = \bigoplus_{m\in\mathbb{N}} \bigoplus_{\lambda\in\mathbb{Z}} \mathrm{gr}^\lambda_{F_v} H^0(X, mL).$$

Hence, $\mathcal{X}_0$ is Proj of the associated graded ring of $F_v^\bullet$. We claim the graded ring is a domain. To see this, observe that if $0 \neq \overline{s} \in \mathrm{gr}^\lambda_{F_v} H^0(X, mL)$ and $0 \neq \overline{t} \in \mathrm{gr}^\mu_{F_v} H^0(X, qL)$, then $v(st) = v(s) + v(t) = \lambda + \mu$ and, hence,

$$0 \neq \overline{s} \cdot \overline{t} \in \mathrm{gr}^{\lambda+\mu}_{F_v} H^0(X, (m + q)L).$$

Therefore, $\mathcal{X}_0$ is integral. Additionally, 1.21.1 implies $\mathcal{X}$ is a normal test configuration of X by Proposition 1.21.3. Using that

$$\mathcal{L} := \tfrac{1}{r}\mathcal{O}_{\mathcal{X}}(1) = -K_{\mathcal{X}/\mathbb{A}^1} + c\mathcal{X}_0$$

for some $c \in \mathbb{Q}$ by Lemma 2.5, $(\mathcal{X}, -K_{\mathcal{X}/\mathbb{A}^1})$ is a normal test configuration of X as well. Thus, (1) and (2) hold.

[2]The minus sign arises from the fact that the equivariant birational map

$$\mathcal{X}|_{\mathbb{A}^1\setminus 0} \to X \times \mathbb{A}^1$$

sends $(x, t) \mapsto (\sigma(t^{-1}, x), t)$.

We sketch the proof of (3). By Proposition 2.6,

$$F^{\lambda}_{\mathcal{X},\mathcal{L}} H^0(X, mL) = F^{\lambda+mrC}_{v_{\mathcal{X}_0}} H^0(X, mL)$$

for some constant C and all $(m, \lambda) \in \mathbb{N} \times \mathbb{Z}$. On the other hand, since $(\mathcal{X}, \mathcal{L})$ is the test configuration constructed using $F_v^{\bullet}$, Remark 1.20 implies that, after replacing L by a power,

$$F^{\lambda}_{\mathcal{X},\mathcal{L}} H^0(X, mL) = F^{\lambda}_{v} H^0(X, mL)$$

for all $(m, \lambda) \in \mathbb{N} \times \mathbb{Z}$. We leave it as an exercise to use the previous two equalities to deduce that $v = v_{\mathcal{X}_0}$. □

PROOF OF 2.7. If $(\mathcal{X}, -K_{\mathcal{X}/\mathbb{A}^1})$ is a test configuration of X with $\mathcal{X}_0$ integral, then $v_{\mathcal{X}_0}$ is dreamy by Proposition 2.6.3. The map is surjective by Lemma 2.9, while the injectivity can be deduced from Proposition 2.6 and the correspondence between test configurations and filtrations. □

3. Valuative criterion

3.1. Fujita's β-invariant. The following invariant of a valuation was developed in the work of Kento Fujita [**Fuj16**, **Fuj19**] and also the work of Chi Li [**Li17b**].

DEFINITION 3.1 (Fujita's β-invariant). Let X be a klt Fano variety and r a positive integer such that $-rK_X$ is Cartier. For a a $\mathbb{Z}$-valued valuation v of klt Fano variety X,

$$\beta_X(v) := A_X(v) - S(v),$$

where $S(v)$ is the limit

$$S(v) = \limsup_{m\to\infty} \frac{\sum_{\lambda\in\mathbb{Z}} \lambda \dim \mathrm{gr}^{\lambda}_{F_v} H^0(X, -mrK_X)}{m \dim H^0(X, -mrK_X)}.$$

As will be discussed in a future section, the previous limsup is actually a limit. If $v := \mathrm{ord}_E$ for some divisor E over X, we simply write

$$\beta_X(E) = A_X(E) - S(E)$$

for the above values.

The motivation for the definition arises from the following computation.

PROPOSITION 3.2. *If $(\mathcal{X}, -K_{\mathcal{X}/\mathbb{A}^1})$ is a test configuration of a klt Fano variety X and $\mathcal{X}_0$ integral, then*

$$\mathrm{Fut}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{A}^1}) = A_X(v_{\mathcal{X}_0}) - S(v_{\mathcal{X}_0}) = c(A_X(E) - S(E)),$$

where $v = c \cdot \mathrm{ord}_E$.

PROOF. To mininimize notation, let $v := v_{\mathcal{X}_0}$ and $A := A_X(v_{\mathcal{X}_0})$. By applying Theorem 2.6 and then Proposition 2.7,

$$\mathrm{Fut}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{A}^1}) = -\frac{(-K_{\mathcal{X}/\mathbb{A}^1})^{n+1}}{(n+1)(-K_X)^n} = -F_0.$$

Above, F_0 is the Laurent expansion

$$\frac{\mathrm{wt} H^0(\mathcal{X}_0, -mK_{\mathcal{X}/\mathbb{A}^1}|_{\mathcal{X}_0})}{m \dim H^0(X, -mK_X)} = F_0 + F_1 m^{-1} + F_2 m^{-2} + \cdots ,$$

where equality holds for $m > 0$ sufficiently divisible. Now, note that

$$\begin{aligned}
\mathrm{wt} H^0(\mathcal{X}_0, -mK_{\mathcal{X}/\mathbb{A}^1}|_{\mathcal{X}_0}) &= \sum_{\lambda \in \mathbb{Z}} \lambda \dim \mathrm{gr}^{\lambda}_{F_{\mathcal{X}, -K_{\mathcal{X}/\mathbb{A}^1}}} H^0(X, -mK_X) \\
&= \sum_{\lambda \in \mathbb{Z}} \lambda \dim \mathrm{gr}^{\lambda + mA}_{F_v} H^0(X, -mK_X) \\
&= \sum_{\lambda \in \mathbb{Z}} (\lambda - mA) \dim \mathrm{gr}^{\lambda}_{F_v} H^0(X, -mK_X) \\
&= \sum_{\lambda \in \mathbb{Z}} \lambda \dim \mathrm{gr}^{\lambda}_{F_v} H^0(X, -mK_X) - mA_X(v_{\mathcal{X}_0}) \dim H^0(X, -mK_X),
\end{aligned}$$

where first equality is by Proposition 1.13 and the second Proposition 2.6. Thus,

$$-F_0 = A_X(v_{\mathcal{X}_0}) - \lim_{m \to \infty} \frac{\lambda \dim \mathrm{gr}^{\lambda}_{F_v} H^0(X, -mrK_X)}{m \dim H^0(X, -mrK_X)}$$

and, hence,

$$\mathrm{Fut}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{A}^1}) = A_X(v_{\mathcal{X}_0}) - S(v_{\mathcal{X}_0}), = c\left(A_X(E) - S(E)\right),$$

where the second equality uses that both terms are homogeneous of degree 1. □

A key feature of the formula in Proposition 3.2 is that it involves only the first term of the weight polynomial of the test configuration induced by $v_{\mathcal{X}_0}$. The second term of the weight polynomial is much more difficult to understand and compute.

PROPOSITION 3.3. *Let X be a klt Fano variety of dimension n. If $\mu : Y \to X$ is a proper birational morphism with Y regular and $E \subset Y$ a prime divisor, then*

$$S(E) = \frac{1}{(-K_X)^n} \int_0^{\infty} \mathrm{vol}(-\mu^* K_X - tE)\, dt$$

REMARK 3.4. While the above integral is from 0 to ∞, it can actually be computed in a finite region, since $\mathrm{vol}(-\mu^* K_X - tE) = 0$ for $t \gg 0$. To see the latter statement, fix an ample divisor A on Y. Since $E \cdot A^{n-1} > 0$, it follows that

$$(-\mu^* K_X - tE) \cdot A^{n-1} = -\mu^* K_X \cdot A^{n-1} - tE \cdot A^{n-1} < 0$$

for $t \gg 0$. Hence, for such t, no positive multiple of $-\mu^* K_X - tE$ has sections.

PROOF. To reduce notation, let $F^\bullet$ denote the filtration of $\bigoplus_{m\in\mathbb{N}} H^0(X,-mrK_X)$ induced by ord_E. Observe that

$$\begin{aligned}\sum_{\lambda\in\mathbb{Z}}\lambda \dim \mathrm{gr}_F^\lambda H^0(X,-mrK_X) &= \sum_{\lambda\in\mathbb{Z}}\lambda(F^\lambda H^0(X,-mrK_X)) - F^{\lambda+1}H^0(X,-mrK_X)) \\ &= \sum_{\lambda=1}^{\infty} F^\lambda H^0(X,-mrK_X) \\ &= \sum_{\lambda=1}^{\infty} \dim H^0(Y,-\mu^* mrK_X-\lambda E) \\ &= \int_0^\infty \dim H^0(Y,-\mu^*mrK_X-\lceil t\rceil E)\,dt \\ &= (mr)^{-1}\int_0^\infty \dim H^0(Y,-\mu^*mrK_X-\lceil t/(mr)\rceil E)\,dt\end{aligned}$$

Thus,

$$S(E)=\limsup_{m\to\infty}\int_0^\infty \frac{\dim H^0(Y,-\mu^*mrK_X-\lceil t/(mr)\rceil E)}{\dim H^0(-mrK_X)}\,dt$$

Since the fraction inside the integral is always ≤ 1 and

$$\lim_{m\to\infty}\frac{\dim H^0(Y,-\mu^*mrK_X-\lceil t/(mr)\rceil E)}{\dim H^0(-mrK_X)}=\frac{\mathrm{vol}(Y,-\mu^*K_X-tE)}{\mathrm{vol}(-K_X)},$$

where the fact that the limit exists follows from [**KK12**, Corollary 3.11], the dominated convergence theorem implies

$$S(E)_X=\int_0^\infty \frac{\mathrm{vol}(Y,-\mu^*K_X-tE)}{\mathrm{vol}(-K_X)}\,dt.$$

$\square$

EXAMPLE 3.5. Using the previous proposition, Fujita's β-invariant can be computed in many examples.

(1) Let $H\subset\mathbb{P}^2$ be a hyperplane. Then $A_{\mathbb{P}^2}(H)=1$ and

$$\begin{aligned}S_{\mathbb{P}^2}(H) &= \frac{1}{-K_{\mathbb{P}^2}^2}\int_0^\infty \mathrm{vol}(-K_{\mathbb{P}^2}-tH)\,dt \\ &= \frac{1}{\mathcal{O}_{\mathbb{P}^2}(3)^2}\int_0^\infty \mathrm{vol}((3-t)H)\,dt = \frac{1}{9}\int_0^\infty (3-t)^2\,dt = 1.\end{aligned}$$

Hence, $A_{\mathbb{P}^2}(H)-S_{\mathbb{P}^2}(H)=1$.

Another way to compute this value is to note that ord_H is induced by a product test configuration $(\mathcal{X},-K_{\mathcal{X}/\mathbb{A}^1})$ of $\mathbb{P}^2$. Hence,

$$A_{\mathbb{P}^2}(E)-S_{\mathbb{P}^2}(E)=\mathrm{Fut}(\mathcal{X},-K_{\mathcal{X}/\mathbb{A}^1})=0,$$

where the last equality uses that $\mathbb{P}^2$ is K-semistable and the test configuration is a product.

(2) Let $B_p\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$ denote the blowup of $\mathbb{P}^2$ at p, $E \subset B_p\mathbb{P}^2$ the exceptional divisor, and L the pullback of a line on $\mathbb{P}^2$. Note that

$$K_{B_p\mathbb{P}^2} = \pi^* K_{\mathbb{P}^2} + E = -3L + E.$$

Thus, we can compute

$$\begin{aligned} S_{B_p\mathbb{P}^2}(E) &= \frac{1}{(-K_{B_p\mathbb{P}^2})^2}\int_0^\infty \mathrm{vol}(-K_X - tE)\,dt \\ &= \frac{1}{(3L-E)^2}\int_0^\infty \mathrm{vol}(3H - E - tE)\,dt = \frac{1}{8}\int_0^2 9-(1+t)^2\,dt = 7/6. \end{aligned}$$

(Note that the above value differs from $S_{\mathbb{P}^2}(E)$.) Therefore,

$$A_{B_p\mathbb{P}^2}(E) - S_{B_p\mathbb{P}^2}(E) = 1 - 7/6 = -1/6 < 0.$$

Since ord_E is dreamy by 2.4, it induces a test configuration with $\mathrm{Fut} = -1/6$. Hence, $B_p\mathbb{P}^2$ is not K-stable.

(3) Generalize the first example, if D is a prime divisor on a klt Fano variety X with $D \sim_{\mathbb{Q}} -cK_X$, then

$$\begin{aligned} S_X(H) &= \frac{1}{(-K_X)^n}\int_0^\infty \mathrm{vol}(-K_X - tH)\,dt \\ &= \frac{1}{(-K_X)^n}\int_0^\infty \mathrm{vol}(-(1-ct)K_X)\,dt = \int_0^{1/q}(1-qt)^n\,dt = \frac{1}{c(n+1)} \end{aligned}$$

3.2. Valuative criterion. The following theorem provides a characterization of K-stability in terms of valuations.

THEOREM 3.6. *Let X be a klt Fano variety. The following hold:*

$$\begin{aligned} X \text{ is K-stable} \quad &\Leftrightarrow \quad A_X(E) - S_X(E) > 0 \text{ for all divisors } E \text{ over } X \\ X \text{ is K-semistable} \quad &\Leftrightarrow \quad A_X(E) - S_X(E) \geq 0 \text{ for all divisors } E \text{ over } X \end{aligned}$$

Additionally, X is K-polystable if and only if $A_X(E) - S(E) \geq 0$ for all divisors E over X and $= 0$ only when ord_E is a product valuation.

Recall, that the term *product valuation* appearing in criterion for K-polystability refers to valuations arising from product test configurations. See Example 2.12 for the precise defintion.]

The K-semistable version of the result was first proven by Kento Fujita [**Fuj19**] and Chi Li [**Li17b**]. The K-stable and K-polystable version requires some slight additional input from [**BX19**].

The theorem implies that we can use divisorial valuations rather than test configurations to understand the K-stability of Fano varieties. This is useful since divisorial valuations can be understood using tools from higher dimensional geometry and, in particular, the Minimal Model Program.

REMARK 3.7 (Dreamy version). We have not yet built up the machinery to prove Theorem 3.6. What we can prove is the following result:

$$X \text{ is K-stable} \quad \Leftrightarrow \quad A_X(E) - S_X(E) > 0 \text{ for all dreamy divisors } E \text{ over } X$$
$$X \text{ is K-semistable} \quad \Leftrightarrow \quad A_X(E) - S_X(E) \geq 0 \text{ for all dreamy divisors } E \text{ over } X$$

Additionally, X is K-polystable if and only if $A_X(E) - S(E) \geq 0$ for all dreamy divisors E over X and $= 0$ only when ord_E is a product valuation.

To see the above statement holds, note that Theorem 1.6 implies that to check K-(poly/semi)stability of X, it suffices to consider the value of the Futaki invariant of test configurations with integral central fiber. Hence, the correspondence between test configurations with integral central fiber and dreamy valuations (Proposition 2.7), as well as the relation between Fut and $A - S$ (Proposition 3.2) imply the above statement.

REMARK 3.8 (Proof of Theorem 3.6). The above remark provides a proof of the reverse implication of Theorem 3.6. Proving the forward implication requires additional machinery. We outline a couple approaches.

(1) Assume X is K-semistable and E is a divisor over X. In order to show $A_X(E) - S_X(E) \geq 0$, we must relate E to a test configuration. If $F^{\bullet}_{\mathrm{ord}_E}$ is not finitely generated, then it does not immediately induce a test configuration.

 To remedy this, one could approximate $F^{\bullet}_{\mathrm{ord}_E}$ by a sequence of finitely generated filtrations $(F^{\bullet}_i)$, which induces test configurations $(\mathcal{X}_i, \mathcal{L}_i)$ and try to show
$$A_X(E) - S_X(E) \geq \lim \mathrm{Fut}(\mathcal{X}_i, \mathcal{L}_i) \geq 0.$$
 Unfortunately, seems quite difficult to approach due to the subtle nature of the b_1 coefficient in the weight polynomial.

 Instead, Fujita shows that a similar approach works if one replaces Fut with the Ding invariant [**Fuj19**]. The Ding invariant of test configuration was introduced by Berman [**Ber16**] and involves a certain log-canonical threshold, rather than the b_1 coefficient. Hence, it is easier to work with.

(2) An alternative approach is to study which divisors over a Fano variety are dreamy. In [**BLX19**, **BLZ19**], it is shown (without using test configurations or the definition of K-stability) that if
$$A_X(E) - S_X(E) < 0$$
 for some divisor E over X, then the inequality holds for some dreamy divisor as well. We will dicsucss this approach in a future section.

3.3. Application to volume bounds. We now discuss an application of the valuative criterion to the volume of Kähler-Einstein Fano varieties. Recall, the *volume* of an n-dimensional klt Fano variety X is the number
$$\mathrm{vol}(X) := \mathrm{vol}(-K_X) := (-K_X)^n.$$

It is > 0 by the assumption that $-K_X$ is ample. Note that

$$\mathrm{vol}(\mathbb{P}^n)) := \mathrm{vol}(\mathcal{O}_{\mathbb{P}^n}(n+1)) = (n+1)^n.$$

REMARK 3.9. It is natural to ask what values the volume can take.

(1) In dimension up to three, there is a classification of smooth Fano varieties. The list shows that if X is a smooth Fano variety of dimension $n \leq 3$ has

$$\mathrm{vol}(X) \leq (n+1)^n$$

and if equality holds then $X \simeq \mathbb{P}^n$.

(2) In higher dimensions, $\mathbb{P}^n$ does not have the largest volume. See for example [**IP99**, p. 128].

(3) By boundedness results of Kollar, Miyaoka, and Mori [**KMM92**], there exists a constant $c(n)$ such that if X is a smooth Fano variety of dimension n, then

$$\mathrm{vol}(X) \leq c(n)^n.$$

By the examples in [**Deb01**, Proposition 5.22], $c(n)$ cannot be chosen to be a polynomial.

As observed by Kento Fujita, the volume of a K-semistable Fano variety is very well behaved.

THEOREM 3.10. *If X is a K-semistable klt Fano varietiy of dimension n, then*

$$\mathrm{vol}(X) \leq (n+1)^n$$

and if equality holds then $X \simeq \mathbb{P}^n$

The result was first established in [**Fuj18**], but with the equality part only proven when X is smooth. The equality part in the singular case was later shown in [**LZ18**].

The result was originally conjectured by differential geometers, Berman and Berndtson, who proved the result for toric varieties [**BB17**]. Differential geometry methods prove a weaker bound than $(n+1)^n$ for Káhler-Einstein Fano varieties [**Deb01**, Section 5.8]. Hence, it was suprising at the time that [**Fuj18**] was written that algebraic methods could establish the optimal bound.

PROOF OF INEQUALITY IN THEOREM 3.10. To prove the inequality, we use the valuative criterion for K-semistability. Fix a smooth point $p \in X$ and let

$$Y := B_pX \xrightarrow{\mu} X$$

denote the blowup of X at p with exceptional divisor $E \subset Y$.

We seek to estimate $S_X(E)$. Consider the short exact sequence

$$0 \to \mathcal{O}_X(-mK_X) \otimes \mathfrak{m}_p^{\lceil mt \rceil} \to \mathcal{O}_X(-mK_X) \to \mathcal{O}_X(-mK_X) \otimes (\mathcal{O}_X/\mathfrak{m}_p^{\lceil mt \rceil}) \to 0,$$

where m is a positive integer such that $-mK_X$ is Cartier and $t \in \mathbb{Q}_{>0}$. Taking H^0 induces an exact sequence

$$0 \to H^0(X, \mathcal{O}_X(-mK_X) \otimes \mathfrak{m}_p^{\lceil mt \rceil}) \to H^0(X, \mathcal{O}_X(-mK_X)) \to H^0(X, \mathcal{O}_X/\mathfrak{m}_p^{\lceil mt \rceil}),$$

where we are using that $\mathcal{O}_X/\mathfrak{m}_p^{\lceil mt \rceil}$ is supported p. Hence,

$$\dim H^0(X, \mathcal{O}_X(-mK_X) \otimes \mathfrak{m}_p^{\lceil mt \rceil}) \geq \dim H^0(X, \mathcal{O}_X(-mK_X)) - \dim(\mathcal{O}_X/\mathfrak{m}_p^{\lceil mt \rceil}).$$

Now, note that

$$H^0(X, \mathcal{O}_X(-mK_X) \otimes \mathfrak{m}_p^{\lceil mt \rceil}) = H^0(Y, \mathcal{O}_Y(-m\mu^* K_X - \lceil mt \rceil E))$$

and

$$\lim_{m\to\infty} \frac{\dim \mathcal{O}_X/\mathfrak{m}^{\lceil mt \rceil}}{m^n/n!} = \lim_{m\to\infty} \left(\frac{\dim \mathcal{O}_X/\mathfrak{m}^{\lceil mt \rceil}}{\lceil mt \rceil^n/n!} \frac{\lceil mt \rceil^n}{m^n} \right) = \operatorname{mult}(\mathfrak{m}_p) t^n = t^n.$$

Above, $\operatorname{mult}(\mathfrak{m}_p) := \lim_{m\to\infty} n! \dim(\mathcal{O}_X/\mathfrak{m}_p^m)$ denote the Hilbert-Samuel multiplicity of the ideal $\mathfrak{m}_p$ [**Har77**, Exc V.3.4] and equals 1 by the asumption that $p \in X$ is a smooth point. Thus, we get

$$\begin{aligned} \operatorname{vol}(-\mu^* K_X - tE) &= \lim_{m\to\infty} \frac{\dim H^0(Y, \mathcal{O}_Y(-m\mu^* K_X - \lceil mt \rceil E))}{m^n/n!} \\ &\geq \lim_{m\to\infty} \left(\frac{\dim H^0(X, \mathcal{O}_X(-mK_X))}{m^n/n!} - \frac{\dim \mathcal{O}_X/\mathfrak{m}^{\lceil mt \rceil}}{m^n/n!} \right) \\ &\geq \operatorname{vol}(X) - t^n. \end{aligned}$$

Now, we can compute

$$\begin{aligned} S(E) = \frac{1}{\operatorname{vol}(X)} \int_0^\infty \operatorname{vol}(-\mu^* K_X - tE) \\ &\geq \frac{1}{\operatorname{vol}(X)} \int_0^{\operatorname{vol}(X)^{1/n}} (\operatorname{vol}(X) - t^n)\, dt = \frac{n}{n+1} \operatorname{vol}(X)^{1/n}. \end{aligned}$$

Finally, observe that

$$n = A_X(E) \geq S(E) \geq \frac{n}{n+1} \operatorname{vol}(X)^{1/n},$$

where the first inequality is by the assumption that X is K-semistable and the valuative criterion (Theorem 3.6). Thus, $\operatorname{vol}(X) \leq (n+1)^n$ as desired. ☐

REMARK 3.11 (Proof of the equality case). To prove the equality case Fujita shows that if $\operatorname{vol}(X) = (n+1)^n$, then

$$\varepsilon_p(-K_X) := \sup\{t \in \mathbb{Q}_{>0} \mid -\mu^* K_X - tE \text{ is nef}\} = (n+1),$$

where $\varepsilon_p(-K_X)$ is the Seshadri constant of $-K_X$ at p [**Laz04**, Section 4.1]. By [**Bau09**, **LZ18**], the latter equality implies $X \simeq \mathbb{P}^n$.

CHAPTER 6

K-stability of Fano varieties and anti-canonical divisors

In this chapter, we discuss two invariants that measure the singularities of anti-canonical divisors on a Fano variety. The first is Tian's alpha-invariant [**Tia87**], which provides a criterion for K-stability. The next, is the δ-invariant (also known as the stability threshold), wich characterizes K-stability and plays an important role in many recent advances in the K-stability of Fano varieties.

1. Invariants of filtrations

Before discussing the α- and δ-invariants, it will be helpful to discuss some invariants of filtrations. Throughout, let X be a klt Fano variety of X and r a positive integer such that $-rK_X$ is Cartier. We set

$$R := \bigoplus_{m\in r\mathbb{N}} R_m := \bigoplus_{m\in r\mathbb{N}} H^0(X, \mathcal{O}_X(-mK_X))$$

and $N_m := \dim(R_m)$ to reduced notation.

We previously worked with $\mathbb{Z}$-filtrations of R. It will be convenient to have the following more general notion.

Definition 1.1. An $\mathbb{R}$-*filtration* $F^\bullet$ of R is a collection of vector subspaces $F^\lambda R_m \subset R_m$ for each $\lambda \in \mathbb{R}$ and $m \in r\mathbb{N}$ such that the following hold

(1) $F^\lambda R_m \subset F^\mu R_m$ when $\lambda \geq \mu$
(2) $F^\lambda R_m = R_m$ and $F^\lambda R_m = 0$ for $\lambda \gg 0$
(3) $F^\lambda R_m = \cap_{\mu<\lambda} F^\mu R_m$
(4) $F^\lambda R_m \cdot F^\mu R_q \subset F^{\lambda+\mu} R_{m+q}$.

Example 1.2. The key examples of $\mathbb{R}$-filtrations are the following:

(1) A $\mathbb{Z}$-filtration $F^\bullet$ of R defines an $\mathbb{R}$-filtration of R by setting $F^\lambda R_m := F^{\lceil \lambda \rceil} R_m$. To see (4) holds, one uses that

$$F^{\lceil \lambda \rceil} R_m \cdot F^{\lceil \mu \rceil} R_q \subset F^{\lceil \lambda \rceil + \lceil \mu \rceil} R_{m+q} \subset F^{\lceil \lambda+\mu \rceil} R_{m}..$$

(2) Any valuation $v \in \mathrm{Val}_X$ induces an $\mathbb{R}$-filtration $F_v^\bullet$ by setting

$$F_v^\lambda R_m = \{ s \in R_m \,|\, v(s) \geq \lambda \}.$$

One reason for working with $\mathbb{R}$-filtrations, is that the data of a real valuation v on X is more naturally encoded in an $\mathbb{R}$-filtration.

The *associated graded ring* of an $\mathbb{R}$ filtration $F^\bullet$ of R is the graded ring

$$\mathrm{gr}_F^\bullet R := \bigoplus_{m\in r\mathbb{N}} \bigoplus_{\lambda\in\mathbb{R}} \mathrm{gr}_F^\lambda R_m,$$

where $\mathrm{gr}_F^\lambda R_m = F^\lambda R_m / F^{>\lambda R_m}$. Here, $F^{>\lambda} R_m = \cup_{\mu>\lambda} F^\mu R_m$. Note that if $F^\bullet$ arises from a $\mathbb{Z}$-filtration (Example 1.2), then $F^{>\lambda} R_m = F^{\lambda+1} R_m$ and, hence, $\mathrm{gr}_F^\bullet R$ agrees with our previous definition.

The *jumping numbers* of $F^\bullet$ is the non-dicreasing sequence of real numbers

$$a_{m,1} \leq \cdots \leq a_{m,N_m}$$

defined by

$$a_{m,j} := \sup\{\lambda \in \mathbb{R} \,|\, F^\lambda R_m \geq j\}.$$

We set

$$S_m(F) := \frac{1}{mN_m} \sum_{j=1}^{N_m} a_{m,j} \quad \text{ and } \quad T_m(F) := \frac{1}{m} a_{m,N_m}$$

equal to the normalized average and maximal jumping numbers. Since $\dim \mathrm{gr}_F^\lambda R_m = \#\{j \,|\, a_{m,j} = \lambda\}$, it follows that

$$S_m(F) = \frac{1}{mN_m} \sum_{\lambda\in\mathbb{R}} \lambda \dim(\mathrm{gr}_F^\lambda R_m)$$

Additionaly, we set

$$S(F) := \limsup_{m\to\infty} S(F) \quad \text{ and } T(F) := \sup_{m\in r\mathbb{N}} T_m(F).$$

We say $F^\bullet$ is *linearly bounded* if there exists an integer $C > 0$ such that $F^{mC} R_m = 0$ for all $m \in r\mathbb{N}$. [1] This is equivalent to the condition that $T(F) < +\infty$.

PROPOSITION 1.3. *If $F^\bullet$ is linearly bounded, then*

$$S(F) := \lim_{m\to\infty} S(F) \quad \textit{and } T(F) := \lim_{m\to\infty} T(F)$$

and the values are finite.

PROOF. Using the multiplicative property of filtration, it follows that

$$mT_m(F) + qT_m(F) \leq (m+q)T_{m+q}(F).$$

Thus, Fekete's Lemma implies the sequence converges.

The proof of the convergence of $S_m(F)$ is similar to the proof of Proposition 3.3. See [**BJ20**, Lemma 2.9] for a proof. □

We will often be interested in the case when $F^\bullet = F_v^\bullet$ for some valuation $v \in \mathrm{Val}_X$. $S_m(v)$, $S(v)$, $T_m(v)$, $T(v)$ for these values. Additionally, when $v = \mathrm{ord}_E$ for some divisor E over X, we $S_m(E)$, $S(E)$, $T_m(E)$, and $T(E)$. We say a valuation v is of linear growth if $T(v) < \infty$ (equivalently, $F_v^\bullet$ is linearly bounded).

[1] There always exists $C > 0$ such that $F^{-mC} R_m = R_m$ for all $m \in r\mathbb{N}$. Indeed, this can be deduced from (2) in the definition of an $\mathbb{R}$-filtration and that R is finitely generated.

Lemma 1.4. *If $v \in \mathrm{Val}_X$ is divisorial, quasi-monomial, or, more generally, $A_X(v) < \infty$, then v is of linear growth.*

Proof. If v is divisorial, then the argument in the proof of ? applies. If v is quasi-monomial, then there exists a divisorial valuation w such that $c_X(v) = c_X(w)$ and $w(f) \geq v(f)$ for all $f \in \mathcal{O}_{X,c_X(v)}$. Since $T(v) \leq T(w)$, the result for divisorial valuations implies v is of linear growth. The case when $A_X(v) < \infty$ follows from an Izumi estimate; see [BJ20]. □

2. Tian's α-invariant

2.1. Definition. Let X be a klt Fano variety and r a positive integer such that $-rK_X$ is Cartier. The *α-invariant* of X is the value

$$\alpha(X) := \inf\{\mathrm{lct}(X, D) \mid 0 \leq D \sim_{\mathbb{Q}} -K_X\},$$

where the infimum runs through all effective $\mathbb{Q}$-divisors $\mathbb{Q}$-linearly equivalent to $-K_X$. Roughly, this invariant measures the singularities of the most singular anti-canonical divisor of X.

It will often be convenient to write

$$\alpha(X) = \inf_{m \in r\mathbb{Z}_{>0}} \alpha_m(X),$$

where

$$\alpha_m(X) := \inf\{\mathrm{lct}(X, \tfrac{1}{m}B) \mid B \in |-mK_X|\}.$$

The equality between the two deinitions follows from the simple observe that if $0 \leq D \sim_{\mathbb{Q}} -K_X$, then $0 \leq D \sim -mK_X$ for some $m > 0$.

Remark 2.1 (History). The α-invariant of a smooth Fano variety was first introduced by Tian in [Tia87] using an analytic definition (i.e. not the one above). There it was shown that if

$$\alpha(X) \geq \frac{\dim(X)}{\dim(X)+1}, \tag{2.1}$$

then X admits a Kähler-Einstein metric.[2] Later, it was shown by Demailly and Kollár that Tian's definition agrees with the one stated above [DK01, CS08]

Example 2.2. We provide a few examples of the α-invariant of a Fano variety.

(1) It is an easy exercise to show $\alpha(\mathbb{P}^1) = \frac{1}{2}$.
(2) Projective space has $\alpha(\mathbb{P}^n) = \frac{1}{n+1}$. Note that while $\mathbb{P}^n_{\mathbb{C}}$ admits a Kähler-Einstein metric, it does not satisfy (2.1).

To compute $\alpha(\mathbb{P}^n) = \frac{n}{n+1}$, first observe that

$$\alpha(\mathbb{P}^n) \leq \mathrm{lct}(\mathbb{P}^n, (n+1)H) = \frac{1}{n+1},$$

[2] An algebraic version of this statement was later proven in [OS12].

where $H \subset \mathbb{P}^n$ is a hyperplane. To verify the reverse inequality, fix a $\mathbb{Q}$-divisor $0 \leq D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n}$. By [Kol97, Lemma 8.10.1],

$$\mathrm{lct}_p(\mathbb{P}^n, D) \geq \frac{1}{\mathrm{ord}_p(D)},$$

where lct_p denotes the lct in a sufficiently small open neighborhood of p. Since $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n}$, $\mathrm{ord}_p(D) \leq n+1$ at any point $p \in \mathbb{P}^n$. Hence,

$$\mathrm{lct}(\mathbb{P}^n, D) = \inf_{p \in \mathbb{P}^n} \mathrm{lct}_p(\mathbb{P}^n, D) \geq \frac{1}{n+1}.$$

Since $0 \leq D \sim_{\mathbb{Q}} -K_X$ was arbitrary, this shows $\alpha(\mathbb{P}^n) \geq \frac{1}{n+1}$.

(3) For a smooth cubic surface $X \subset \mathbb{P}^3$, it was computed in [Che08] that

$$\alpha(X) = \begin{cases} 2/3 & \text{if } X \text{ contains an Eckhardt point} \\ 3/4 & \text{otherwise} \end{cases},$$

where an Eckhardt point is a point $0 \in X$ that is the intersection of three lines on the surface (a general cubic surface will not have such point). This example shows that the α-invariant encodes information on the goemetry of a Fano variety.

(4) If $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of deg $d \leq n+1$, then it is easy to see

$$\alpha(X) \leq \frac{1}{n+2-d}.$$

Indeed, choose a general hyperplane $H \subset \mathbb{P}^n$. Since $\mathcal{O}_X(-K_X) = \mathcal{O}(n+2-d)$, $(n+2-d)H|_X \sim_{\mathbb{Q}} -K_X$ and, hence,

$$\alpha(X) \leq \mathrm{lct}(X, (n+2-d)H|_X) \leq \frac{1}{n+2-d}.$$

Some of the above examples show a deficiency of the use of the α-invariant in studying K-stability. For example, while every smooth projective hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is expected to admit a Kähler-Einstein metrics,

$$\alpha(X) < \frac{n}{n+1},$$

when $d \leq n$. Hence, Tian's criterion (2.1) cannot be used to verify the conjecture.

2.2. Valuative interpretation. We now explain that Tian's α-invariant can be interpreted in terms of the log discrepancy and T-invariant of a valuation. Indeed, for any valuation $v \in \mathrm{Val}_X$, note that

$$T_m(v) = \sup\{\lambda \,|\, F^{m\lambda} R_m \neq 0\} = \frac{1}{m} \sup\{v(D) \mid D \in |-mK_X|\}$$

and, hence,

$$T(v) = \sup\{v(D) \,|\, 0 \leq D \sim_{\mathbb{Q}} -K_X\}.$$

Remark 2.3 (Relation to pseudo-effective threshold). If E is a prime divisor on a normal variety Y with a proper birational morphism $\mu : Y \to X$, then $T(E) > t$ if and only if

$$H^0(Y, \mathcal{O}_Y(-m(\mu^* K_X - tE)m)) \neq 0$$

for some $m > 0$. Hence, $T(E)$ is the threshold

$$T(E) := \sup\{t \in \mathbb{Q}_{>0} \mid -\mu^* K_X - tE \text{ is pseudo-effective}\}.$$

See [**Laz04**] for the definition of pseudo-effective.

The following formulas were shown in [**Amb16**, pg. 10][**BJ20**, Section 4.1], and follow easily from the definitions.

Proposition 2.4. *If X is a klt Fano variety, then*

$$\alpha_m(X) = \inf_E \frac{A_X(E)}{T_m(E)} = \inf_v \frac{A_X(v)}{T_m(v)}$$

and

$$\alpha(X) = \inf_E \frac{A_X(E)}{T(E)} = \inf_v \frac{A_X(v)}{T(v)},$$

where where the first pair of infimums run through all divisors E over X and the second through all valuations v of linear growth.

Proof. For the first equality, observe that

$$\alpha_m(X) = \inf_{D \in |-mK_X|} \operatorname{lct}(X, \tfrac{1}{m}D) = \inf_{D \in |-mK_X|} \inf_E \frac{A_X(E)}{\operatorname{ord}_E(\frac{1}{m}D)}$$
$$= \inf_E \inf_{D \in |-mK_X|} \frac{A_X(E)}{\operatorname{ord}_E(\frac{1}{m}D)} = \inf_E \frac{A_X(E)}{T_m(E)}.$$

Since $\operatorname{lct}(X, D) = \inf_{v \in \operatorname{Val}_X} \frac{A_X(v)}{v(D)}$ by Remark 1.19, we also know $\operatorname{lct}(X, D) = \inf_v \frac{A_X(v)}{v(D)}$, where infimum runs through all valuations of linear growth. Repeating the same argument as above shows the second equality for $\alpha_m(X)$ holds.

The equalities for $\alpha(X)$ can be deduced from the expressions for $\alpha_m(X)$. Indeed,

$$\alpha(X) := \inf_{m \in r\mathbb{N}} \alpha_m(X) = \inf_{m \in r\mathbb{N}} \inf_E \frac{A_X(E)}{T_m(E)} = \inf_E \sup_{m \in r\mathbb{N}} \frac{A_X(E)}{T_m(E)} = \inf_E \frac{A_X(E)}{T(E)}$$

and the proof of the second equality is similar. □

3. Stability threshold

In [**FO18**], Fujita and Odaka defined a modified version of the α-invariant in the hopes of characterizing K-stability. Their approach was to measure the singularities of certain special anti-canonical divisors, rather than all of them.

3.1. Definition.

DEFINITION 3.1. A $\mathbb{Q}$-divisor $0 \leq D \sim_{\mathbb{Q}} -K_X$ is *m-basis type* if there exists a basis $s_1, \ldots, s_{N_m}$ of $H^0(X, \mathcal{O}_X(-mK_X))$ such that

$$D := \frac{1}{mN_m} (\{s_1 = 0\} + \cdots + \{s_{N_m} = 0\}) ..$$

Note that normalization factor by $\frac{1}{mN_m}$ is so that the resulting divisors is $\mathbb{Q}$-linearly equivalent to $-K_X$.

As we will see in Proposition 3.5, m-basis type divisors are related to the S-invariant, which appears in the valuative criterion for K-stability. This relation is the motivatin for considering such divisors.

EXAMPLE 3.2. On $\mathbb{P}^n$, the divisor

$$D = H_0 + \cdots + H_{n+1},$$

where there $H_i = \{x_i = 0\}$ are the coordinate hyperplanes, is an m-basis type divisor for all $m > 0$. To see this choose the basis $s_1, \ldots, s_{N_m}$ for

$$H^0(X, \mathcal{O}_X(-mK_X)) \simeq H^0(X, \mathcal{O}_X(m(n+1))$$

given by the monomials in $x_0, \ldots, x_{n+1}$ of degree $m(n+1)$ and consider the m-basis type divisor $D := \frac{1}{mN_m} \sum_i \{s_i = 0\}$. Since $\mathrm{Supp}(s_j = 0)$ is contained in the union of the coordinate hyperplanes,

$$D = b_0 H_0 + \cdots b_n H_n.$$

for some rational numbers $b_1, \ldots, b_n$. Since the construction is symmetric in the x_i variables, $b_0 = \cdots = b_n$. Using that $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n}$, we then conclude each $b_i = 1$.

The stability threshold (also, known as the δ-invariant) is defined by measuring the singularities of m-basis type divisors.

DEFINITION 3.3 (Stability threshold). [FO18] For each $m \in r\mathbb{N}$, set

$$\delta_m(X) := \inf\{\mathrm{lct}(X, D) \,|\, 0 \leq D \sim_{\mathbb{Q}} -K_X \text{ is } m\text{-basis type}\}$$

and

$$\delta(X) := \limsup_{m \to \infty} \delta_m(X).$$

In general, it is quite difficult to compute the stability threshold using only the definition (and not additional machinery that has been developed). We begin with the simplest example.

EXAMPLE 3.4. We claim that $\delta_m(\mathbb{P}^1) = 1$ for all $m \in \mathbb{N}$. Hence, $\delta(\mathbb{P}^1) = 1$.

To verify the claim, first note recall that $\{0\} + \{\infty\}$ is an m-basis type divisor on $\mathbb{P}^1$ by the previous example. Thus,

$$\delta_m(\mathbb{P}^1) \leq \mathrm{lct}(\mathbb{P}^1, \{0\} + \{\infty\}]) = 1.$$

For the reverse inequality, fix a basis $s_0, \ldots, s_{2m}$ of

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2m)) = \bigoplus_{i=0}^{2m} k \cdot x^i y^{2m-i}$$

and set

$$D := \frac{1}{m(2m+1)}\{s_0 \cdots s_{2m} = 0\}.$$

To show $\mathrm{lct}(\mathbb{P}^1, D) \geq 1$, it suffices to show $\mathrm{ord}_p(D) \leq 1$ at each point $p \in \mathbb{P}^1$. After changing coordinates, we may assume $p = \{x = 0\} \in \mathbb{P}^1$. After reordering our basis, we may assume

$$\mathrm{ord}_p(s_0) \geq \ldots \geq \mathrm{ord}_p(s_{2m}).$$

Since $\langle s_0, \ldots, s_j \rangle$ are linearly independent and $\mathrm{ord}_P(s_j)$ is the lowest degree of x appearing as a monomial in s_j,

$$\mathrm{ord}_p(s_j) \geq 2m - j$$

Thus,

$$\mathrm{ord}_p(D) = \frac{1}{m(2m+1)} \sum_{i=0}^{2m} \mathrm{ord}_p(s_i) \leq \frac{1}{m(2m+1)} \sum_{i=0}^{2m} (2m - i) = 1.$$

as desired. Therefore, $\mathrm{lct}(\mathbb{P}^1, D) \geq 1$.

The computation in the previous paragraph generalizes to give a formula for S-invariant in terms of m.

Proposition 3.5. *If v is a valuation on X and $m \in r\mathbb{N}$, then*

$$S_m(v) := \max\{v(D) \mid D \text{ is } m\text{-basis type}\}$$

and the max is achieved if D arises from a basis diagonalizing $F_v^\bullet$.

A basis $s_1, \ldots, s_{N_m}$ is said to diagonalize an $\mathbb{R}$-filtration if each $F^\lambda R_m$ is spanned by some subset of basis elements. Equivalently, if we set $\lambda_i := \sup\{\lambda \in \mathbb{R} \mid s \in F^\lambda R_m\}$, the condition means that

$$F^\lambda R_m = \langle s_i \mid \lambda_i \geq \lambda \rangle$$

for each $\lambda \in \mathbb{R}$. Note that such a basis always exists.

Proof. Let $s_1, \ldots, s_{N_m}$ be basis for R_m and set $D := \frac{1}{mN_m}\{s_1 \ldots s_{N_m} = 0\}$. After rearranging the indices, we may assume $v(s_1) \geq \cdots \geq v(s_{N_m})$. Thus, for any $1 \leq j \leq N_m$, $\langle s_1, \ldots, s_j \rangle \subset F^{v(s_j)} R_m$ and, hence, $a_{m,j} \geq v(s_j)$. Now, we compute

$$v(D) = \frac{1}{mN_m} \sum_{j=1}^{N_m} v(s_j) \leq \frac{1}{mN_m} \sum_{j=1}^{N_m} a_{m,j} = S_m(v).$$

Additionally, if $s_1, \ldots, s_{N_m}$ diagonalizes the filtration, then the above inequalities are equalities and we get $v(D) = S_m(v)$. □

The previous proposition implies the following valuative formula for $\delta_m(X)$.

PROPOSITION 3.6. *If X is a klt Fano variety, then*

$$\delta_m(X) = \inf_E \frac{A_X(E)}{S_m(E)} = \inf_v \frac{A_X(v)}{S_m(v)},$$

where the first infimum runs through divisors E over X and the second through valuations v of linear growth.

PROOF. We compute

$$\delta_m(X) = \inf_{D\ m\text{-basis}} \operatorname{lct}(X,D) = \inf_{D\ m\text{-basis}} \inf_E \frac{A_X(E)}{\operatorname{ord}_E(D)}$$
$$= \inf_E \inf_{D\ m\text{-basis}} \frac{A_X(E)}{\operatorname{ord}_E(D)} = \inf_E \frac{A_X(E)}{S_m(E))}$$

where the last equality is by Proposition 3.5. The second equality can be deduced as in the proof of Proposition 2.4. □

3.2. Characterization of stability. The importance of the δ-invariant is its relationship to the valuative criterion for K-stability.

THEOREM 3.7. [**BJ20**, Theorem B] *If X is klt Fano variety, then*

$$\delta(X) = \inf_E \frac{A_X(E)}{S_m(E)} = \inf_v \frac{A_X(v)}{S_m(v)},$$

where the first infimum runs through divisors E over X and the second through valuations v of linear growth. Additionally, $\delta(X) := \lim_{m\to\infty} \delta_{mr}(X)$.

Combining the previous theorem with the valuative criterion for K-stability implies the following.

COROLLARY 3.8. [**FO18**,**BJ20**] *If X is a K-semistable Fano variety, then*

$$X \text{ is K-semistable} \iff \delta(X) \geq 1.$$

The above corollary was originally conjectured in [**FO18**], where the reverse impication was shown. The result was completed in [**BJ20**]. The subtlety in proving Theorem 3.7, and, hence, deducing its corollary, is in understanding the convergence of the relevant invariants. The key result one needs, is the following technical lemma, which we state without proof.

PROPOSITION 3.9. [**BJ20**, Corollary 2.10] *If X is a klt Fano variety and $\varepsilon > 0$, then there exists $m_0 := m_0(\varepsilon, X)$ such that*

$$S_m(v) \leq (1+\varepsilon)S(v)$$

for all $m \geq m_0$ and $v \in \operatorname{Val}_X$ of linear growth.

Note that in the above statement, the value m_0 is independent of the choice of v. The result is shown using the theory of Okounkov bodies in [**LM09**].

Proof of Theorem 3.7. We will only prove the first equality, since the proof of the second is similar. Since $\delta_m(X) = \inf_E \frac{A_X(E)}{S_m(E)}$ by Proposition 3.6 and $S(E) = \lim_{m\to\infty} S_m(E)$,

$$\limsup_{m\to\infty} \delta_m(X) \le \inf_E \frac{A_X(E)}{S(E)}.$$

Next, fix $\varepsilon > 0$. By Proposition 3.9, there exists m_0 such that $S_m(v) \le (1+\varepsilon)S(v)$ for all $m \ge m_0$ and $v \in \mathrm{Val}_X$ of linear growth. Hence,

$$\liminf_{m\to\infty} \delta_m(X) = \liminf_{m\to\infty} \inf_E \frac{A_X(E)}{S_m(E)} \ge \frac{1}{1+\varepsilon} \inf_E \frac{A_X(E)}{S(E)},.$$

Sending $\varepsilon \to 0$ implies $\liminf_{m\to\infty} \delta_m(X) \ge \inf_E \frac{A_X(E)}{S(E)}$. Therefore, $\lim_{m\to\infty} \delta_m(X) = \inf_E \frac{A_X(E)}{S(E)}$ as desired. □

3.3. Computing examples. Given that the stability threshold characterizes K-stability, there has been great intersest in computing it in examples.

Example 3.10. We list a few examples of the value of the stability threshold

(1) For toric varieties, the stability threshold has a simple formula in terms of the barycenter of the anti-canonical polytope P_{-K_X} as shown in [**BJ20**]. Using this, one can compute
   (a) $\delta(\mathbb{P}^n) = 1$,[3]
   (b) $\delta(B_p\mathbb{P}^2) = 6/7$, and
   (c) $\delta(B_{p,q}\mathbb{P}^2) = 21/25$.
   Note that $B_p(\mathbb{P}^2)$ and $B_{p,q}\mathbb{P}^2$ are the two examples of K-unstable del Pezzo varieties. Hence, it makese sense that there δ-invariants are < 1.
(2) For a smooth cubic surface $X \subset \mathbb{P}^3$, it was computed in [**AZ21**] that

$$\alpha(X) = \begin{cases} 3/2 & \text{if } X \text{ contains an Eckhardt point} \\ 27/17 & \text{otherwise} \end{cases},$$

Remark 3.11. It can be quite subtle to compute the δ invariant in a given example. In computations, one often leverages the following results:

(1) If $\delta(X) < 1$, then

$$\delta(X) = \inf_E \frac{A_X(E)}{S(E)}$$

where the infimum runs through all $\mathrm{Aut}(X)$-equivariant divisors E over X such that ord_E is $\mathrm{Aut}(X)$-equivariant [**Zhu21**]. When X has a large automorphism group, there may be very few $\mathrm{Aut}(X)$ equivariant divisors. For example, there are no $\mathrm{Aut}(\mathbb{P}^n)$ equivariant divisors over $\mathbb{P}^n$!

[3] The $n = 2$ first computed in [**PW18**] by carefully analyzing the lct's of m-basis type divisors on $\mathbb{P}^2$.

(2) In [**AZ21**], the authors develop a technique for computing the δ-invariant by restricting to lower dimensional subvarieties using certain special admissible flags. This technique was used heavily in [**ACC+21**] to comupte which smooth Fano threefolds are K-stable.
(3) By [**BLX19**], $\min\{1, \delta(X)\}$ is lower semicontinuous and constructible families. Hence, to show a general member of a family of Fano varieties is K-sesmistable, it suffices to show a single one is K-semistable.

3.4. Properties. We now discuss various properties of the stability threshold, which shows it is a natural invariant for studying K-stability.

3.4.1. *Analytic interpretation.* In [**Ber21**, **CRZ19**], it was shown that if X is a smooth complex Fano variety, then

$$\min\{1, \delta(X)\} = \sup\{c \in [0,1] \,|\, \exists\, \omega \in c_1(X) \text{ such that } \mathrm{Ric}(\omega) > cw\}.$$

The quantity on the right hand side is the greatest Ricci lower bound of X and can be regarded of as a measure of how close one come to constructing a Kähler-Einstein metric on X. It appeared in the work of Tian [**Tia92**] and was studied by Székelyhidi [**Szé11**].

3.4.2. *Volume bounds.* Using the stability threshold, Fujita's volume bound for K-semistable Fano varieties admits a natural generalization to the K-unstable case.

THEOREM 3.12. *If X is a klt Fano variety of dimension n, then*

$$\mathrm{vol}(X)^{1/n} \leq (n+1)\delta(X)^{-1}$$

and equality holds if and only if $X \simeq \mathbb{P}^n$.

PROOF. Following the proof of Theorem 3.7, pick a smooth point $p \in X$, and let $E \subset B_pX \xrightarrow{\mu} X$ denote the exceptional divisor of the blowup of X at p. We compute

$$\mathrm{vol}(X)^{1/n} \leq \frac{(n+1)}{n} S_X(E) = (n+1)\frac{S_X(E)}{A_X(E)} \leq (n+1)\delta(X)^{-1},$$

where the first inequality shown in the proof of Theorem 3.7 and the second inequality is by Theorem 3.7.

It remains to analyze the case when equality holds. If $\mathrm{vol}(X)^{1/n} = (n+1)\delta(X)^{-1}$, the previous equation implies $S_X(E) = n+1$. Now, [**Fuj18**, Theorem 2.3] implies

$$\varepsilon_p(-K_X) := \sup\{t \,|\, -\mu^*K_X - tE \text{ is nef}\} = n+1.$$

Finally, [**LZ18**, Theorem 2] implies $X \simeq \mathbb{P}^n$. □

3.4.3. *Relation to α-invariant.* The following theorem compares the stability threshold to Tian's α-invariant.

THEOREM 3.13. [**BJ20**, Theorem C] *If X is a klt Fano variety of dimension n, then*

$$\frac{n+1}{n}\alpha(X) \leq \delta(X) \leq (n+1)\alpha(X).$$

An immediate consequence of the above theorem combined with Theorem 3.6 and 3.7 is the following algebraic version of Tian's criterion for K-stability.

COROLLARY 3.14. *If X is a klt Fano variety of dimension n and*

$$\alpha(X) \geq \frac{n}{n+1} \qquad \textit{resp.,} \ > \frac{n}{n+1},$$

then X is K-semistable (resp., K-stable).

The above corollary was first proven algebraically in [**OS12**] using the original definition of K-stability. In [**FO18**], it was realized that the statement could be deduced as an easy consequence of the valuative criterion for K-stability. In [**Fuj19**], it was shown that if X is a smooth Fano variety of dimension $n \geq 2$ with $\alpha(X) = \frac{n}{n+1}$, then X is K-stable.

In order to prove Theorem 3.13, we first prove an inequality between the S and T-invariants.

PROPOSITION 3.15. *If X is a klt Fano variety and $v \in \mathrm{Val}_X$ a valuation of linear growth, then*

$$\frac{1}{n+1}T(v) \leq S(v) \leq \frac{1}{n+1}T(v).$$

To prove the second inequality above, we use a clever approach of [**AZ21**], which relies on the following simple linear algebra statement whose proof we leave for the reader. For different approaches, see [**Fuj19**, **FO18**].

LEMMA 3.16. [**AZ21**, Lemma 3.1] *If $F^\bullet$ and $G^\bullet$ are $\mathbb{R}$-filtrations of R and $m \in r\mathbb{N}$, then there exists a basis $s_1, \ldots, s_{N_m}$ of R_m that diagonalizes both $F^\bullet$ and $G^\bullet$.*

PROOF OF PROPOSITION 3.15. We only prove the second inequality, which is the one needed to verify Tian's criterion. For a proof of the first inequality, see [**BJ20**, Lemma 2.6].

Fix an integer $q > 0$ such that $-qK_X$ is a very ample. Hence, a general element $H \in |-qK_X|$ will be a prime divisor and satisfy $c_X(v) \notin H$. Now, by Lemma 3.16, we may choose a basis $s_1, \ldots, s_{N_m}$ that is compatible with both $F_E^\bullet$ and $F_H^\bullet$, and consider the m-basis type divisor $D := \frac{1}{mN_m}\sum_i \{s_i = 0\}$. By Lemma 3.5,

$$S_m(H) = \mathrm{ord}_H(D) \quad \text{ and } \quad S_m(v) = v(D).$$

(This m-basis type divisor is quite exotic in that it vanishes to a high degree along two very different valuations!) Since $\mathrm{ord}_H(D) = \mathrm{coeff}(H)$, we can write

$$D = S_m(H)H + \Gamma,$$

where Γ is an effective $\mathbb{Q}$-divisor. Note that

$$\Gamma = D - S_m(H)H \sim_{\mathbb{Q}} -K_X - S_m(H)q(-K_X) \sim_{\mathbb{Q}} -(1 - qS_m(H))K_X$$

Next, we compute

$$S_m(v) = v(D) = S_m(H)v(H) + v(\Gamma) = v(\Gamma)$$
$$= (1 - qS_m(H))v((1 - qS_m(H))^{-1}\Gamma) \leq (1 - qS_m(H))T(v),$$

where the third equality uses that $c_X(v) \notin H$, and the inequaulity uses that $0 \leq (1 - qS_m(H))^{-1}\Gamma \sim_{\mathbb{Q}} -K_X$. Hence, sending $m \to \infty$ gives

$$S(v) \leq (1 - qS(H))T(v) = \frac{n}{n+1}T(v),$$

where the equality is Example 3.5.3. □

Proof of Theorem 3.13. The result follows immedatiately from Theorem 3.7 and Propositions 2.4 and 3.15. □

4. Complements

In this section, we use the language of complements to define special classes of divisorial valuations that satisfy finite generation properties and, hence, induce test configurations. This leads to a proof of the forward implication of the valuative criterion for K-stability (Theorem 3.6).

4.1. Complements. The theory of complements was introduced by Shokurov in the study of 3-fold log flips [**Sho92**] and goes back to earlier work on anti-canonical divisors on Fano threefolds [**Sho79**].

Definition 4.1. Let X be a klt Fano variety. A *complement* of X is a $\mathbb{Q}$-divisor Δ such that (X, Δ) is lc and $K_X + \Delta \sim_{\mathbb{Q}} 0$. We say Δ is an *N-complement*, where $N \in \mathbb{Z}_{>0}$, if additionally, $N(K_X + \Delta) \sim 0$.

Informally, a complement is a way to turn a Fano variety into a log Calabi-Yau pair.

Example 4.2. A 1-complement on $\mathbb{P}^2$ is simply a divisor $\Delta \in |-K_{\mathbb{P}^2}|$ such that $(\mathbb{P}^2, \Delta)$ is lc. Hence, Δ can be either

(1) a smooth cubic,
(2) a nodal cubic,
(3) a conic intersecting a line transversely, or
(4) three lines not all intersecting at a point.

Remark 4.3 (Birkar's boundedness). An important result of Birkar [**Bir19**] states that for each integer $n > 0$, there exists an integer $N := N(n) > 0$ such that the following holds: If X is a klt Fano variety of dimension n, then it admits an N-complement. A version of this result will play a key role in a later section.

4.2. Log canonical places.

Definition 4.4. An *lc place* of a pair (X,Δ) is a valuation $v \in \mathrm{Val}_X$ such that $A_{X,\Delta}(v) = 0$. When $v = \mathrm{ord}_E$ for some divisor E over X, then we simply say E is an lc place of (X,Δ).

Informally, the lc places are valuations that make the pair fail to be klt. In this case when $v = \mathrm{ord}_E$, the condition that $A_{X,\Delta}(E) = 0$ simply means

$$\mathrm{coeff}_E(K_Y - \mu^*(K_X + \Delta)) = -1,$$

where E arises from the data of $E \subset Y \xrightarrow{\mu} X$. Equivalently, $\mathrm{coeff}_E(\Delta_Y) = 1$, where $K_Y + \Delta_Y = f^*(K_X + \Delta)$.

Definition 4.5. If X is a klt Fano variety and E is a divisor over X, we say E is an *lc place of a complement* if there exist a complement Δ of X such that E is an lc place of (X,Δ).

Example 4.6. Continuing with Example 4.2, let us consider lc places of 1-complements on $\mathbb{P}^2$.

(1) If $C \subset \mathbb{P}^2$ is a smooth cubic curve, the only lc place of $(\mathbb{P}^2, C)$ is ord_C.
(2) For the complement $(\mathbb{P}^2, \{xyz = 0\})$, the lc places are the toric valuations.

4.3. Relation to test configurations. In this section, we show that lc places of complements induces test configurations and analyze the geometry of such degenerations.

Proposition 4.7. *Let X be a klt Fano variety and E a divisor over X. If E is an lc place of a complement, then E is dreamy.*

Recall, dreamy means that the filtration induced by E of $R := \bigoplus_{m\in\mathbb{N}} H^0(X, -rK_X)$ is finitely generated. Equivalently, the k-algebra

$$\bigoplus_{(m,p)\in\mathbb{N}\times\mathbb{Z}} H^0(Y, -m\mu^* rK_X - pE)$$

is finitely generated, where E arises from the data of $E \subset Y \xrightarrow{\mu} X$. By Lemma 2.9, this implies E induces a test configuration $(\mathcal{X}, -K_{\mathcal{X}/\mathbb{A}^1})$ of X with $\mathcal{X}_0$ integral.

As we will see, the result is consequence of finite generation results in [**BCHM10**].

Proof. Choose a complement Δ on X such that E is an lc place of (X,Δ). Fix a log resolution $\mu : W \to X$ of (X,Δ) such that E arises a prime divisor $E \subset W$. Since (X,Δ) is lc and $A_{X,\Delta}(E) = 0$, there exists $0 < \varepsilon < 1$ such that

$$(X, (1-\varepsilon)\Delta) \text{ is klt} \quad \text{ and } \quad A_{X,(1-\varepsilon)\Delta}(E) < 0.$$

Hence, [**BCHM10**, Corollary 1.4.3] implies there exists a proper birational morphism $\mu : Y \to X$ such that Y is $\mathbb{Q}$-factorial, E is a divisor on Y, and $\mathrm{ExcDiv}(\mu) \subset E$. (Note that the inclusion could be strict if E is a prime divisor on X.)

Now, define a $\mathbb{Q}$-divisor Γ on Y by the formula

$$K_Y + \Gamma = \mu^*(K_X + (1-\varepsilon)\Delta).$$

Note that Γ is effective, since $\mathrm{ExcDiv}(\mu) \subset E$ and $\mathrm{coeff}_E(\Delta) = 1 - A_{X,\varepsilon\Delta}(E) < 1$. Thus, the condition that $(X, (1-\varepsilon)\Delta)$ is klt implies (Y, Γ) is klt. Additionally, $-K_Y - \Gamma$ is big and nef, since

$$-K_Y - \Gamma \sim_{\mathbb{Q}} -\varepsilon f^*(K_X),$$

and $-K_X$ is ample. Therrefore, Y is a Mori dream space by [**BCHM10**, Corollary 1.3.1]. Hence, the multi-graded section ring

$$\bigoplus_{(m,p)\in\mathbb{N}\times\mathbb{Z}} H^0(Y, -m\mu^* r K_X - pE)$$

must be finitely generated. □

The previous proposition implies lc places of complements induce test configurations. The following theorem shows which test configurations those are.

THEOREM 4.8. [**BLX19**] *Let $(\mathcal{X}, -K_{\mathcal{X}/\mathbb{A}^1})$ be a test configuration of a klt Fano variety X. The following are equivalent.*

(1) The pair $(\mathcal{X}, \mathcal{X}_0)$ is lc.

(2) The divisorial valuation $v_{\mathcal{X}_0} := r(\mathrm{ord}_{\mathcal{X}_0})$ is an lc place of a complement.

PROOF. **Add proof** □

4.4. Delta invariant and lc places of complements.

PROPOSITION 4.9. *If X is a klt Fano variety of dimension n with $\delta(X) \leq \frac{n+1}{n}$, then*

$$\delta(X) = \inf_E \frac{A_X(E)}{S(E)},$$

where the infimum runs through all divisors E over X that are lc places of complements.

The theorem was first proven in [**BLZ19**, **BLX19**] when $\delta(X) \leq 1$ and improved to the $\delta(X) < \frac{n+1}{n}$ case in [**LXZ22**] using Lemma 3.16.

PROOF. By Theorem 3.7, we know the inequality "$\leq$" between the two sides of the equation holds. To prove the reverse inequality, fix $0 < \varepsilon \ll 1$. By Theorem 3.7 and Proposition 3.9, there exists $m \in r\mathbb{N}$ such that

(1) $\delta_m(X) \leq (1+\varepsilon)\delta(X)$ and

(2) $S_m(v) \leq (1+\varepsilon)S(E)$ for all valuations v on X of linear growth.

By Proposition 3.6, there exists a divisor E over X such that $\delta_m(X) = \frac{A_X(E)}{S_m(E)}$. Hence,

$$\frac{A_X(E)}{S(E)} \leq (1+\varepsilon)\frac{A_X(E)}{S_m(E)} = (1+\varepsilon)\delta_m(X) \leq (1+\varepsilon)^2\delta(X).$$

We claim that E is also an lc place of a complement when m is sufficiently large. Assuming the claim, the "$\geq$" in the statement of the proposition holds, which completes the proof.

To verify the claim, fix an integer $q > 0$ such that $|-qK_X|$ is very ample and choose a general divisor $H \in |-qK_X|$ such that H is irreducible and $c_X(E) \notin H$. By Lemma 3.16, there exists a basis $\{s_1, \dots s_{N_m}\}$ for $H^0(X, -mK_X)$ that diagonalizes both filtrations $F_E^\bullet$ and $F_H^\bullet$. Consider the induced m-basis type divisor

$$D := \frac{1}{mN_m}\big(\{s_1 = 0\} + \cdots + \{s_{N_m} = 0\}\big).$$

By Proposition 3.5,

$$S_m(E) = \mathrm{ord}_E(D) \quad \text{and} \quad S_m(H) = \mathrm{ord}_H(D) = \mathrm{coeff}_H(D).$$

The latter implies, we can write

$$D = D' + S_m(H)H$$

for some effective $\mathbb{Q}$-divisor D'. Since $c_X(E) \notin H$, $\mathrm{ord}_E(D) = \mathrm{ord}_E(D')$. Now, observe

$$\delta_m(X) \leq \mathrm{lct}(X, D) \leq \mathrm{lct}(X, D') \leq \frac{A_X(E)}{\mathrm{ord}_E(D')} = \frac{A_X(E)}{S_m(E)} = \delta_m(X).$$

Thus, above inequalities are equalities. Therefore, $(X, \delta_m D')$ is lc and

$$A_{X,\delta_m D}(E) = A_X(E) - \delta_m \mathrm{ord}_E(D') = 0.$$

Finally, observe that

$$\delta_m D' = \delta_m(D - S_m(H)H) \sim_{\mathbb{Q}} -c_m K_X,$$

where $c_m := \delta_m(X)(1 - kS_m(H))$. Using Example 3.5, we see

$$\lim_{m\to\infty} c_m = \delta(X)\,(1 - kS(H)) < 1$$

Hence, after possibly increasing m, we may assume $c_m < 1$. Since H is general, if we set

$$\Delta := \delta_m(X)D' + (1 - c_m)q^{-1}H,$$

then (X, Δ) is lc, $A_{X,\Delta}(E) = 0$, and $\Delta \sim_{\mathbb{Q}} -K_X$. Therefore, Δ is a complement with lc place E. □

Using a similar style of argument, one can show that divisorial minimizers of the stability thresholds are lc places of complements.

Proposition 4.10. [**BLX19**, **LXZ22**] *If X is a klt Fano variety and E a divisor over X such that*

$$\delta(X) = \frac{A_X(E)}{S(E)} < \frac{n+1}{n},$$

then E is an lc place of a complement.

4.5. Relation to valuative criterion. Using the results in the previous section, we can now finish the proof of the valuative criterion for K-stability.

Proof of Theorem 3.6. As we saw in Remark 3.7,

$$X \text{ is K-stable} \quad \Leftrightarrow \quad A_X(E) - S_X(E) > 0 \text{ for all dreamy divisors } E \text{ over } X$$
$$X \text{ is K-semistable} \quad \Leftrightarrow \quad A_X(E) - S_X(E) \geq 0 \text{ for all dreamy divisors } E \text{ over } X$$

Additionally, X is K-polystable if and only if $A_X(E) - S(E) \geq 0$ for all dreamy divisors E over X and $= 0$ only when ord_E is a product valuation.

To prove the forward implication in the K-semistable case, we argue by contradiction. If

$$A_X(E) - S(E) < 0$$

for some divisor E over X, then $\delta(X) < 1$ by Theorem 3.7. Hence, Proposition 4.7 and 4.9 implies there exists a dreamy divisor E over X such that

$$A_X(E) - S(E) < 0.$$

Therefore, the above paragraph implies X is not K-semistable, which completes the proof in the K-semistable case.

To complete the proof in the K-stable and K-polystable case, note that if X is K-semistable and E a divisor over X such that $A_X(E) - S(E) = 0$, then $1 = \delta(X) = A_X(E)/S(E)$. Hence, Proposition 4.10 implies E is dreamy. Therefore, the result in the K-stable and K-polystable case follows from the K-semitable case and the first paragraph. $\square$

CHAPTER 7

K-moduli of Fano varieties

In this chapter, we survey recent progress on constructing moduli spaces parametrizing K-polystable Fano varieties.

1. Moduli theory in algebraic geometry

2. Moduli of Fano varieties

3. K-moduli of Fano varieties

The K-moduli theorem asserts that K-polystable klt Fano varieties with fixedd numerical invariatns are parametrized by a projective moduli space. The precise statement, which uses the language of algebraic stacks, is stated here.

THEOREM 3.1 (K-moduli). *Fix an integer $n > 0$ and a rational number $v > 0$.*

(1) *(Moduli Stack) There exists a finite type Artin stack*

$$\mathcal{M}_{n,v}^{\mathrm{Kss}}$$

parametrizing families of K-semistable klt Fano vartieties of dimension n and voume v.

(2) *(Moduli space) There exists a morphism to a good moduli space*

$$\mathcal{M}_{n,v}^{\mathrm{Kss}} \to M_{n,v}^{\mathrm{Kps}},$$

which is a projective scheme parametrizing K-polystable klt Fano varieties of dimension n and volume v.

REMARK 3.2 (History). The above theorem was proven in various stages.

(1) (Del Pezzo surfaces) Odaka, Spotti, and Sun constructed compactifications of the moduli space of smooth Kähler-Einstein del Pezzo surfaces by adding in singular Kähler-Einstein Fano varieties at the boundary [**OSS16**]. (The degree 4 del Pezzo case was previously completed by Mabuichi and Mukai in [**MM93**]). The constructions are explicit and rely on differential geometry input. The authors conjectured that a similar statement should hold more generally [**OSS16**, Conjecture 6.2]

(2) (Smoothable Fano varieties) Li, Wang, and Xu [**LWX19**] (see also [**Oda15a**]) constructed a good moduli space parametrizing smoothable K-polystable Fano varieties of fixed dimension and volume. The construction relies heavily on differential geometric input, in pariticular, Kähler-Einstein metrics and

results of Donaldsonand Sun on Gromov-Hausdorff limits of Kähler-Einstein Fano varieties [DS14].

(3) (General case) Using recent progress in understanding the K-stability of Fano varieties, purely algebraic methods were used to prove Theorem 3.1. This was acheived in a long list of papers by authors including Alper, Blum Codogni, Halpern-Leistner, Jiang, Li, Liu, Patakfalvi, Wang, Xu, Zhuang with the final step completed in a papper of Liu, Xu, and Zhuang.

The proof of Theorem 3.1 requires understanding various properties of K-semitable Fano varieties and their behaviour in families. The proof can be broken down into the following steps.

(1) Definition of moduli stack
(2) Boundedness
(3) Openness of K-semistability
(4) Uniqueness of K-polystable degenerations
(5) Construction of good moduli space
(6) Properness
(7) Projectivity

In the rest of this section, we explain each of these statements.

3.1. Defininiton of moduli stack. The first step in the K-moduli theorem is to precisely define our moduli problem. To do so, we must answer the seemingly elementary question:

What is a family of Fano varieties $X \to S$?

Clearly, $X \to S$ should be a flat proper morphism with fibers that are Fano varieties. A more subtle, but also desirable, requirement is for the anti-canonical $\mathbb{Q}$-line bundles to vary in well behaved manner. This is made precise in the following definition, which is modelled on Kollär's definition of a family of KSBA-stable varieties.

DEFINITION 3.3. A *family of klt Fano varieties* is a flat proper morphism $X \to S$ such that

(1) X_s is a klt Fano variety for all $s \in S$ and
(2) $\omega_{X/S}^{[m]}$ commutes with base change for all $m \in \mathbb{Z}$.

REMARK 3.4 (Kollár's condition). Statement (2) of Definition 3.3 is often referred to as Kollár's condition. As the name suggests, Kollár introduced the condition when defining a family of KSBA-stable varieties [Kol13b, Section 4].

The sheaf in (2) is defined as the push forward

$$\omega_{X/S}^{[m]} := j_* \omega_{U/S}^{\otimes m},$$

where $U \subset X$ is the open smooth locus of $X \to S$ and $j : U \hookrightarrow X$ the natural inclusion. Note that $\operatorname{cod}_X(X \setminus U) \geq 2$. Indeed, this follows from the observation that $(X \setminus U) \cap X_s = \operatorname{Sing}(X_s)$ and $\operatorname{cod}_{X_s}(\operatorname{Sing}(X_s)) \geq 2$, where the latter is by the assumption that X_s is klt and, in particular, normal.

The statement that $\omega_{X/S}^{[m]}$ *commutes with base change* means that, for every morphism $T \to S$, the natural map

$$f_T^*\omega_{X/S}^{[m]} \to \omega_{X_T/T}^{[m]}$$

is an isomorphis, where $f_T : X_T := X \times_S T \to X$ is the pullback of $f : X \to S$.

REMARK 3.5 (Normal base). In the case when S is a normal Noetherian scheme, we can define a *relative canonical divisor* $K_{X/S}$ on X as a divisor satisfying

$$\omega_{U/S} \simeq \mathcal{O}_U(K_{X/S}|_U),$$

where $U \subset X$ is the smooth locus of $X \to S$. Note that

$$\omega_{X/S}^{[m]} \simeq j_*\mathcal{O}_U(mK_{U/S}) \simeq \mathcal{O}_X(mK_{X/S}).,$$

where the last isomorphism is by the normality of X, which follows from the assumption that $X \to S$ is flat with normal fibers and S is normal by [?].

In this setting, Definition 3.3.2 can be replaced by the condition that

(2') The relative canonical divisor $K_{X/S}$ is $\mathbb{Q}$-Cartier.

See [**Kol22**, Theorem 3.1] for the equivalence of the two definitions.

With Definition 3.3 in hand, we can now define our moduli stack.

DEFINITION 3.6 (K-Moduli stack). $\mathcal{M}_{n,v}^{Kss}$ is the fibered category over Sch_k whose

- objects are families of klt Fano varieties $X \to S$ such that X_s is K-semistable of dimension n and volume v for each $s \in S$.
- maps $[X' \to S'] \to [X \to S]$ consist of the data of maps $X' \to X$ and $S' \to S$ such that the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

  is Cartesian.

3.2. Boundedness. The next step is to prove the moduli problem is bounded.

THEOREM 3.7. [**Jia20**, **XZ21**] *The set of klt Fano varieties* $\mathcal{M}_{n,v}^{Kss}(\Bbbk)$ *is bounded.*

The result was first proven by Jiang [**Jia20**] using Birkar's proof of the BAB Conjecture [**Bir21**]. A few years later, Xu and Zhuang [**XZ21**] deduced the result from Hacon, McKernan, and Xu's solution to the Baytrev Conjecture [**HMX14**], which is an input in [**Bir21**].

REMARK 3.8 (Boundedness). The term *bounded* means that there is a projective morphism of finite type k-schemes

$$Y \to S$$

such that the following holds: if $[X] \in \mathcal{M}^{Kss}_{n,v}(\Bbbk)$, then $X \simeq Y_s$ for some $s \in S$. The key is that S is finite type, and, hence, implies that the set of Fano varieties can be embedded in a fixed projective space.

In the above definition, it is equivalent to assume assume that $Y \to S$ is a family of klt Fano varieties. Indeed, by an argument similar to the proof of [**Kol22**, Corollary 3.3], there is a disjoint union of locally closed subsets $i : S' \hookrightarrow S$ such that the base change

$$Y' := Y \times_S S' \to S'$$

is a family of klt Fano varieties and if Y_s is a klt Fano variety, then $s \in i(S')$. Hence, replacing $Y \to S$ with $Y' \to S'$ implies the stronger statement holds.

3.3. Openness of K-semistability. The next step is to show that K-semistability is an open condition in families.

THEOREM 3.9 ([**BLX19**],[**Xu20**]). *If $X \to S$ is a family of klt Fano varieties, then the locus*

$$S^{\mathrm{Kss}} := \{s \in S \,|\, X_s \textit{ is K-semistable}\}$$

is a Zariski open set of S.

There are two proofs of this theorem: [?Xu19] uses local methods and the characterization of K-semistability uses the normalized volume function, while the proof in [**BLX19**] uses global methods and the characterization of K-semistability in terms of the delta invariant.

THEOREM 3.10. [**BLX19**] *If $X \to S$ is a family of klt Fano varieties, then the function $S \to \mathbb{R}$ sending*

$$s \mapsto \min\{1, \delta(X_s)\}$$

is lower semi-continuous and constructible.

Using that $S^{\mathrm{Kss}} = \{s \in S \,|\, \delta(X_s) \geq 1\}$, the previous theorem immediately implies the openess of K-semistability. As well, the K-stable locus is open and the K-polystable locus[1] is constructible by [**BLX19**].

With boundedness and openness complete, a standard argument shows that $\mathcal{M}^{Kss}_{n,v}$ is a disjoint union of finitely many quotient stacks of the form

$$[Z/\mathrm{PGL}_{N+1}],$$

where $Z \subset \mathrm{Hilb}_h(\mathbb{P}^N)$ is a locally closed subset of the Hilbert scheme. This then implies $\mathcal{M}^{Kss}_{n,v}$ is an algebraic stack.

3.4. Uniqueness of degenerations. The next step is to analyze the uniqueness of degenerations of K-semistable Fano varieties. This was done in a series of works.

[1]The K-polystable locus is *not* always open, since there exist strictly K-semistable Fano varieties and Theorem 3.11 states that they admit degenerations via test configurations to K-polystable Fano varieties.

3.4.1. *Degenerations along test configurations.* Li, Wang, and Xu first analyzed degenerations of K-semistable Fano varieties along test configurations.

THEOREM 3.11. [**LWX21**] *If X is a K-semistable klt Fano variety, then*

(1) (Existence) there exists a test configuration degenerating

$$X \rightsquigarrow X_0$$

to a K-polystable klt Fano variety and

(2) (Uniqueness) X_0 is uniquely determined by X.

The strategy to prove Theorem 3.11 is to first prove the following helpful lemma.

LEMMA 3.12. *Let $\mathcal{X}$ be a special test configuration of a K-semistable klt Fano variety X. Then $\mathcal{X}_0$ is K-semistable if and only if* $\mathrm{Fut}(\mathcal{X}) = 0$

PROOF. The forward implication is straightforward. Indeed, the test configuration $\mathcal{X}$ induces a $\mathbb{G}_m$-action on $\mathcal{X}_0$ and, hence, a map $\lambda : \mathbb{G}_m \to \mathrm{Aut}(\mathcal{X}_0)$ satisfying

$$\mathrm{Fut}(\mathcal{X}) = \mathrm{Fut}(\mathcal{X}_0, \lambda),$$

where $\mathrm{Fut}(\mathcal{X}_0, \lambda)$ is the Futaki invariant of the induced product test configuration of $\mathcal{X}_0$. Using that $\mathrm{Fut}(\mathcal{X}_0, \lambda) = -\mathrm{Fut}(\mathcal{X}, \lambda^{-1})$ and the assumption that $\mathcal{X}_0$ K-semistable, we conclude $\mathrm{Fut}(\mathcal{X}_0, \lambda) = 0$.

The proof of the converse is more involved and uses the equivalence between K-stability and $\mathbb{G}_m$-equivariant K-stability; see [**LWX21**, Lemma 3.1] for details. □

Now, to prove Theorem 3.11.1, assume X is K-semistable, but not K-polystable. Then there exists a special test configuration degenerating

$$X \rightsquigarrow X_0$$

such that $X \not\cong X_0$ and $\mathrm{Fut} = 0$. Hence, X_0 is K-semistable by the above lemma. If X_0 is again not K-polystable, we can repeat the above argument

$$X \rightsquigarrow X_0 \rightsquigarrow X_1 \rightsquigarrow \cdots \rightsquigarrow X_r.$$

Using the bounded of K-semistable Fano varieties, it follows that this process must terminate with a K-polystable klt Fano variety X_r. Next, one must show there is a test configuration degenerating $X \rightsquigarrow X_r$, which requires extra work; see [**LWX19**, page 21].

The proof of Theorem 3.11.2 is harder and requires input from the minimal model program. In light of the theorem, it makes sense to introduce the following equivalence relation.

DEFINITION 3.13. (S-equivalence) Two klt Fano varieties X and X' are *S-equivalent* if there exists special test configurations of X and X' degenerating

$$X \rightsquigarrow X_0 \leftsquigarrow X'$$

to a K-polystable klt Fano variety X_0.

REMARK 3.14. This definitions of polystability and S-equivalence are modelled on the notions for vector bundles on a smooth projective curve C.

(1) A vector bundle E on C is polystable if it can be written as a direct sum of stable vector bundles.
(2) If E is a semistable vector bundle on C, then tThe *Jordan-Holdar* filtration $F_{JH}^{\bullet}$ of E has the property that

$$\mathrm{gr}_{F_{JH}}^{\bullet}(E) = \bigoplus_{\lambda \in \mathbb{Z}} F^{\lambda}E/F^{\lambda+1}E$$

is a direct sum of stable vector bundles. Hence, $\mathrm{gr}_{F_{JH}}^{\bullet}(E)$ is polystable.

Note that the Rees construction for $F_{JH}^{\bullet}$ gives a $\mathbb{G}_m$-equivariant family of vector bundles on $\mathcal{E}$ on $C \times \mathbb{A}^1$ such that $\mathcal{E}_1 \simeq E$ and $\mathcal{E}_0 \simeq \mathrm{gr}_{F_{JH}}^{\bullet}(E)$. Thus, we get a degeneration

$$E \rightsquigarrow \mathrm{gr}_{F_{JH}}^{\bullet}(E)$$

from a semistable vector bundle to a polystable vector bundle.
(3) Two semistable vector bundles E and E' on C are *S-equivalent* if

$$\mathrm{gr}_{F_{JH}}^{\bullet}(E) \simeq \mathrm{gr}_{F'_{JH}}^{\bullet}(E').$$

In the moduli space of semistable vector bundles, two vector bundles are identified if and only if they are S-equivalent.

3.4.2. *Degenerations over curves.* Now, the we understand the iniqueness of K-polystable degenerations via test configurations, we can move on to degenerations over more general curves.

THEOREM 3.15. [**BX19**] *Let $0 \in C$ be the germ of a smooth curve and*

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[dl] \\ & C & \end{tikzcd}$$

two families of klt Fano varieties with an isomorphism

$$\phi : X|_{C \setminus 0} \simeq X'|_{C \setminus 0}$$

over $C \setminus 0$.

(1) If X_0 is K-stable and X'_0 is K-semistable, then ϕ extends to an isomorphism $X \simeq X'$ over C.
(2) If X_0 and X'_0 are K-polystable, then $X_0 \simeq X'_0$.
(3) If X_0 and X'_0 are K-semistable, then X_0 are X'_0 are S-equivalent.

Roughly, the statement says that moduli space of K-polystable Fano varieties $M_{n,v}^{Kps}$. To make this precise, one must first construct it!

An easy application of Theorem 3.15.1 is the following corollary.

REMARK 3.16. It follows from previous results that for a point $[X] \in \mathcal{M}^{Kss}_{n,v}(\mathbb{k})$

$$[X] \text{ is closed in } \mathcal{M}^{Kss}_{n,v} \quad \Leftrightarrow \quad X \text{ is K-polystable..}$$

Indeed, Theorem 3.11.1 implies the forward implication, while Theorem 3.15.2 implies the reverse implication.

3.5. Construction of K-moduli space. The next step is to show the existence of a good moduli space parametrizing K-polytable Fano varieties.

THEOREM 3.17. [**ABHLX20**] *There exists a morphism to a good moduli space*

$$\mathcal{M}^{Kss}_{n,v} \to M^{Kps}_{n,v},$$

where $M^{Kps}_{n,v}$ is a separated algebraic space whose k-points are in bijection with isomorphism classes of K-polystable klt Fano varieties of dimension n and volume v.

3.5.1. *Good moduli spaces.* The notion of a good moduli space was introduced by Alper in order to generalize Mumford's notion of a good quotient to the setting of algebraic stacks.

DEFINITION 3.18. [**Alp13**] A *good moduli space* is a morphism

$$\phi : \mathcal{M} \to M$$

from an algebraic stack to an algebraic space such that

(1) ϕ_* is exact on quasi-coherent sheaves
(2) the natural map $\mathcal{O}_M \to \phi_*\mathcal{O}_{\mathcal{M}}$ is an isomorphism.

While the definition is remarkably simple, such maps satisfy many nice properties.

PROPOSITION 3.19. [**Alp13**, Main Properties] *If $\phi : \mathcal{M} \to M$ is a good moduli space, then*

(1) ϕ is surjective and universally closed,
(2) ϕ is initial among maps from $\mathcal{M}$ to algebraic spaces,
(3) $x, x' \in \mathcal{M}(k)$ are identified by $\mathcal{M} \to M$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \emptyset$.

EXAMPLE 3.20. We list a few important examples of good moduli spaces

(1) If $\mathcal{M}$ is a Deligne-Mumford stack and $\phi : \mathcal{M} \to M$ is a coarse moduli space[2], then ϕ is a good moduli space.
(2) If G is a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$, then the morphism
$$[X^{\mathrm{ss}}/G] \to X//G$$
is a good moduli space
(3) If G is a reductive group acting on a finite type $\mathbb{k}$-scheme $\mathrm{Spec}(A)$, then
$$[\mathrm{Spec}(A)/G] \to \mathrm{Spec}(A^G)$$
is a good moduli space.

[2]A map $\mathcal{M} \to M$ is a *coarse moduli space* if it is initial among maps from $\mathcal{M}$ to algebraic spaces and is a bijection on k-points

3.5.2. *Construction of moduli space.* The construction of $M^{Kps}_{n,v}$ relies on a theorem of Alper, Halpern-Lesitner, and Heinloth, which gives a criterion for when an algebraic stack admits a good moduli space [**AHLH18**]. In verifying the condition, one relies on the birational geometry argument from [**BX19**, **LWX21**].

It follows from the construction of $M^{Kps}_{n,v}$ that the good moduli space morphism

$$\mathcal{M}^{Kss}_{n,v} \to M^{Kps}_{n,v}$$

is étale locally at a closed point $[X] \in \mathcal{M}^{Kps}_{n,v}$ of the form

$$[\operatorname{Spec}(A)/G] \to \operatorname{Spec}(A^G),$$

where $G = \operatorname{Aut}(X)$. As part of the proof, the following statement is shown.

THEOREM 3.21. [**ABHLX20**] *If X is a K-polystable Fano variety, then* $\operatorname{Aut}(X)$ *is reductive.*

3.6. Properness. In the previous steps, we constructed $M^{Kps}_{n,v}$ as a separated algebraic space. The next step is to show properness.

THEOREM 3.22. [**LXZ22**] *The algebraic space $M^{Kps}_{n,v}$ is proper.*

To verify $M^{Kps}_{n,v}$ is proper, it suffices to show the stack $\mathcal{M}^{Kss}_{n,v}$ satisfies the existence part of the valuative criterion of properness for DVRs [**AHLH18**, Theorem A].

THEOREM 3.23. *Let R be a DVR with fraction field $K :=$* $\operatorname{Frac}(R)$*.*

If X_K is K-semistable klt Fano variety over K, then there exists a finite extension of DVRS $R \subset R'$ and family of K-semistable klt Fano varieties

$$X' \to \operatorname{Spec}(R')$$

such that $X'_{K'} \simeq X_K \times_K K'$, where $K' :=$ $\operatorname{Frac}(R')$*.*

REMARK 3.24. We make some brief remarks on the proof this difficult result.

(1) In the special case when the base field $k = \mathbb{C}$, the properness of the smoothable locus $\overline{M^{Kps,sm}_{n,v}}$ has analytic proof due Donaldson and Sun. Indeed, if $0 \in C$ is a smooth pointed curve $X \to C \setminus 0$ a family of Kähler-Einstein smooth Fano varieties, then each X_t has the structure of a metric space for $t \neq 0$. Roughly, one can consider the Gromov-Hausdorff limit

$$X_0 := \lim_{t \to 0} X_t,$$

which is a priori just a metric space. By [**DS14**], X_0 in fact has the structure of Kähler-Einstein klt Fano variety.

(2) The proof of Theorem 3.23 by [**LXZ22**] relies on [?BHLLX19], which states that the destabilization conjecture implies Theorem 3.23. Recall, the conjecture, which states that if X is K-unstable klt Fano variety, then

$$\delta(X) = \inf_E \frac{A_X(E)}{S(E)}$$

is a minimum, was verified in [**LXZ22**].

3.7. Projectivity. The last remaining step is to show $M^{\mathrm{Kps}}_{n,v}$ a projective scheme. To do so, the strategy is to exhibit an ample line bundle on the moduli space. There is a natural line bundle to consdier that is motivated by both both algebraic and analytic considerations.

3.7.1. *Chow Mumford line bundle.* The CM line bundle associates to any any family of polarized schemes a line bundle on the base. As we will see, the definiton is closely related to K-stability and the Futaki invariant.

To define the line bundle, fix a flat proper morphism of connected schemes $X \to S$ and $\mathcal{L}$ a relatively ample line bundle on X. Let $n = \dim X - \dim S$.

(1) By a result of Knudsen and Mumford [**MFK94**, pg 184], there exist line bundles $\mathcal{M}_0, \ldots, \mathcal{M}_{n+1}$ on S such that
$$\det(f_*\mathcal{L}^m) = \mathcal{M}_{n+1}^{\binom{m}{n+1}} \otimes \mathcal{M}_n^{\binom{m}{n}} \otimes \cdots \otimes \mathcal{M}_0^{\binom{m}{0}}$$
for all $m \gg 0$.

(2) By the theory of Hilbert polynomials, there are rational numbers $a_0, \ldots, a_n$ such that
$$\chi(X_s, \mathcal{L}_s^m) = a_0 m^n + a_1 m^{n-1} + \cdots + a_n,$$
for all $m \geq 0$. By the assumption that f is flat, this is independent of $s \in S$.

The expansion in (1) may be thought of as a Hilbert function with values in $\mathrm{Pic}(S)$.

Definition 3.25 (Paul-Tian). The CM-line bundle of $f : (X, L) \to S$ is
$$\lambda_{CM,f,L} := \mathcal{M}_{n+1}^{n(n+1)+\mu(L)} \otimes \mathcal{M}_n^{-2(n+1)},$$
where $\mu(L) = 2a_1/a_0$. Note that it is in fact a $\mathbb{Q}$-line bundle.

The CM-line bundle satisfies the follow natural properties.

Remark 3.26. The CM line bundle satisfies the following natural properties.

(1) If we replace L with L^m for some positive integer m, then
$$\lambda_{\mathrm{CM},f,L^m} = \lambda_{\mathrm{CM},f,L}^{m^n}.$$
This follows from a straightforward computation and gives a natural way to define the CM line bundle when L is $\mathbb{Q}$-line bundle.

(2) If $g : T \to S$ is a morphism of schemes, then there is a canonical isomorphism
$$g^*\lambda_{\mathrm{CM},f,L} = \lambda_{CM,f_T,L_T},$$
where $f_T : X_T := X \times_S T \to T$ and L the pullback of L via $X_T \to X$. This follows from functorial properties of the Knudsen-Mumford expansion.

(3) If $(\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ is a test configuration, then
$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) := \frac{a_0}{(n+1)!} \deg \lambda_{\mathrm{CM},\overline{\pi},\overline{\mathcal{L}}},$$
where $\overline{\pi} : (\overline{\mathcal{X}}, \overline{\mathcal{L}}) \to \mathbb{P}^1$ is the compactified test configuration. [3]

[3]To verify the formula, recall that $\det(\overline{\pi}_*\overline{\mathcal{L}}^m) \simeq \mathcal{O}_{\mathbb{P}^1}(w_m)$ for $m \gg 0$ by the proof of Proposition 2.7.

(4) If X and S are normal projective varieties, then

$$c_1(\lambda_{\mathrm{CM},f,L}) = f_*\left(\mu(L)c_1(L)^{n+1} + (n+1)c_1(L_n)\cdot c_1(K_{X/T})\right),$$

where the equality is of rational cycles on S (or equivalently, $\mathbb{Q}$-divisors on S up to linear equivalence). See e.g. [**CP21**, Lemma A.2]

3.7.2. *CM line bundle and Fano varieties.* If $f : X \to S$ is a family of klt Fano varieties of relative dimension n, then we set

$$\lambda_{\mathrm{CM},f} := \frac{1}{r^n}\lambda_{\mathrm{CM},f,L},$$

where r is a positive integer such that $L := \omega_{X/S}^{[-r]}$ is a line bundle. Since the line bundle is functorial, it naturally induces a $\mathbb{Q}$-line bundle λ_{CM} on $\mathcal{M}_{n,v}^{Kss}$. Additional arguments show λ_{CM} desdence to a $\mathbb{Q}$-line bundle L_{CM} on $M_{n,v}^{Kps}$.

In order to verify the ampleness L_{CM}, the key step is to verify the following positivity statements.

THEOREM 3.27. *Let $f : X \to S$ be a family of K-semistable Fano varieties over a projective normal scheme S.*

*(1) (Nefness) [**CP21**] The $\mathbb{Q}$-line bundle $\lambda_{\mathrm{CM},f}$ is nef.*
*(2) (Bigness) [**CP21**,**XZ20**,**LXZ22**] If the moduli map $g : S \to M_{n,v}^{Kps}$ is generically finite, then $\lambda_{\mathrm{CM},f}$ is big.*

Theorem 3.27.2 was proven in three stages. In [**CP21**], the statement was verified when the generic fiber of $X \to S$ is uniformly K-stable, and, in [**XZ20**], under the assumption that K-polystability is equivalent to a notion called uniform reduced uniform K-stability [**XZ20**, Definition 3.4]. In [**LXZ22**], the authors showed (reduced) uniform K-stability is equivalent to K-(poly)stability and, hence, completed the proof of the bigness statement.

Using the Nakai-Moshezon criterion for ampleness and some additional techinical arguments, one can then deduce the following corollary that was acheived in its full form in [**LXZ22**].

COROLLARY 3.28. *The $\mathbb{Q}$-line bundle L_{CM} on $M_{n,v}^{Kps}$ is ample.*

Using that $M_{n,v}^{Kps}$ is a proper algebraic space and the ampleness of L_{CM}, it follows that

COROLLARY 3.29. *The good moduli space $M_{n,v}^{Kps}$ is a projective scheme.*

REMARK 3.30 (Analytic interpretation). The CM line bundle on $M_{n,v}^{Kps}$ is closely related to the Weil-Peterson metric on the locus of $M_{n,v}^{Kps}$ parametrzing smooth Kähler-Einstein Fano varieties. This connection was utilized in [**LWX18**] to show L_{CM} is big and nef on the locus $M_{n,v}^{Kps,sm} \subset M_{n,v}^{Kps}$ parametrizing smooth K-polystable Fano varieties and deduce that $M_{n,v}^{Kps,sm}$ is a quasi-projective variety. The latter was proven prior to the algebraic advances in understanding K-stability.

REMARK 3.31 (Canonically polarized varieties). Similar to the Fano setting, there is induced CM $\mathbb{Q}$-line bundle on the KSBA moduli space $M_{n,v}^{KSBA}$. Its ampleness was verified in [**PX17**] using positivity results from [**KP17**].

3.8. Examples.

CHAPTER 8

Normalized volume

APPENDIX A

Algebraic groups

In this section, we review notions concerning algebraic groups, group actions, and linearized line bundles. This material can be found in [**MFK94**], as well as the lecture notes [**Hos15**].

We include this material, since actions and linearizations of the multiplicative group play a key role in the definition of K-stability. While we mostly only need the case of the multiplicative group, it is helpful to recall the whole theory.

Conventions: Throughout, all schemes are defined over an algebraically closed field k of arbitrary characteristic.

1. Algebraic groups

DEFINITION 1.1. An *algebraic group* G is a scheme with morphisms $m : G \times G \to G$, $i : G \to G$ and $e : \mathrm{Spec}(k) \to G$ such that the following diagrams commute:

(1) ASSOCIATIVITY

$$\begin{tikzcd} G \times G \times G \arrow[r, "{m,\mathrm{id}_G}"] \arrow[d, "{\mathrm{id}_G,m}"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G. \end{tikzcd}$$

(2) IDENTITY

$$\begin{tikzcd} G \arrow[r, "e\times\mathrm{id}_G"] \arrow[dr, "\mathrm{id}_G"'] & G \times G \arrow[d, "m"] & G \arrow[l, "\mathrm{id}_G\times e"'] \arrow[dl, "\mathrm{id}_G"] \\ & G & \end{tikzcd}$$

(3) INVERSES

$$\begin{tikzcd} G \arrow[r, "\mathrm{id}_G\times i"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "i\times\mathrm{id}_G"'] \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "e"] & G & \mathrm{Spec}(k) \arrow[l, "e"'] \end{tikzcd}$$

As a consequence of the definition, $G(k)$ is endowed with the structure of a group.

EXAMPLE 1.2. We list some important examples of algebraic groups. In the below examples G is affine. Hence, defining $m : G \times G \to G$ and $i : G \to G$ amounts to defining the rings maps

$$m^* : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) \quad \text{and} \quad i^* : \mathcal{O}(G) \to \mathcal{O}(G).$$

The choice of $e : \mathrm{Spec}(k) \to G$ will be clear from context.

(1) the *multiplicative group* $\mathbb{G}_m := \mathrm{Spec}(k[t, t^{-1}])$, where

$$m^* t = t \otimes t \quad \text{and } i^* t = t^{-1}.$$

Note that $\mathbb{G}_m(k) = (k^\times, \cdot)$.

(2) the *additive group* $\mathbb{G}_a := \mathrm{Spec} k[t]$, where

$$m^* t = 1 \otimes t + t \otimes 1 \quad \text{and } i^* = -t.$$

Note that $\mathbb{G}_a(k) = (k, +)$.

(3) the *general linear group* $\mathrm{GL}_n = \mathrm{Spec}[x_{i,j} : 1 \leq i, j \leq n]_{\det(x_{i,j})}$. We leave it to the reader to write down m^* and i^* so that $\mathrm{GL_n}(k)$ is the group of invertible $n \times n$-matrices with values in k.

DEFINITION 1.3. A *homomorphism* of algebraic groups $G \to H$ is a morphism of schemes $G \to H$ such that the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{m_G} & G \\ {\scriptstyle f,f}\downarrow & & \downarrow{\scriptstyle f} \\ H \times H & \xrightarrow{m_H} & H \end{array}$$

A *subgroup of an algebraic group G is a closed subscheme $H \subset G$ such that closed embedding $H \hookrightarrow G$ is a homomorphism of algebraic groups.*

EXAMPLE 1.4. Below are a few examples of group homomorphisms and subgroups.

(1) The morphism $\det : \mathrm{GL}_n \to \mathbb{G}_m$ is a homomorphism.

(2) A *linear representation* of an algebraic group G is a homomorphism

$$G \to \mathrm{GL}(V)$$

for some vector space V.

(3) For each integer $\lambda \in \mathbb{Z}$, the morphism $\chi_\lambda : \mathbb{G}_m \to \mathbb{G}_m$, where

$$\chi_\lambda^* : k[t, t^{-1}] \to k[t, t^{-1}] \quad \text{sends} \quad t \mapsto t^\lambda,$$

is a group homomorphism. In fact, all group homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ are of this form.

(4) The group of *n-th roots of unity*

$$\mu_n := \mathrm{Spec}(k[t, t^{-1}]/(t^n - 1)) \subset \mathbb{G}_m$$

is a subgroup of $\mathbb{G}_m$. Note that when $\mathrm{char}(k) = p$, the scheme μ_p is non-reduced!

(5) If G is an affine finite type algebraic group, then G is a subgroup of GL_n for some $n > 0$. [?]

2. Group actions

Definition 2.1. An *action* of an algebraic group G on a scheme X is a morphism $\sigma : G \times X \to X$ such that the following diagrams commute:

$$\begin{tikzcd} G \times G \times X \arrow[r, "{m,\mathrm{id}_X}"] \arrow[d, "{\mathrm{id}_G,\sigma}"] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Spec}(k) \times X \arrow[r, "{e,\mathrm{id}_X}"] \arrow[dr] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

The above data induces a group action $G(k)$ on the set $X(k)$.

Example 2.2. We list two examples of actions of algebraic groups.

(1) The standard action $\mathbb{G}_m$ on $\mathbb{A}^1$ is the action the sends $t \cdot x = tx$ for all $t \in \mathbb{G}_m(k)$ and $x \in \mathbb{A}^1(k)$. This corresponds to the morphism

$$\sigma : \mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{A}^1$$

where $\sigma^* : k[x] \to k[t, t^{-1}] \otimes k[x]$ is defined by $x \mapsto x \otimes t$.

(2) The algebraic group PGL_{n+1} acts on $\mathbb{P}^n$.

If a group G acts on a scheme X, then there is an induced action $G(k)$ on the set of functions $\mathcal{O}(X)$ such that

$$(g \cdot f)(x) = f(g^{-1} \cdot x)$$

for all $g \in G(k)$, $f \in \mathcal{O}_X$, and $x \in X$. The inverse appears so that the action is a left action in the case when G is not abelian.[1]

Example 2.3. We list two examples of this when $G = \mathbb{G}_m$.

(1) For the standard action of $\mathbb{G}_m$ on $\mathbb{A}^1$,

$$a \cdot x^m = a^{-m} x^m$$

for all $a \in \mathbb{G}_m(k)$ and $\mathcal{O}(\mathbb{A}^1) = k[x]$. Note the minus sign, which is produced by the inverse in the above definition.

(2) If $\mathbb{G}_m$ acts on an affine scheme $\mathrm{Spec}(A)$, then A admits a structure of a $\mathbb{Z}$-graded ring $A = \bigoplus_{\lambda \in \mathbb{Z}} A_\lambda$, where

$$A_\lambda := \{f \in A \,|\, a \cdot f = a^\lambda f \text{ for all } a \in \mathbb{G}_m(k)\}.$$

Here the $\bigoplus_{\lambda \in \mathbb{Z}} A_\lambda$ is called the *weight decomposition* and A_λ the λ-*weight space*. Conversely, a $\mathbb{Z}$ grading of a ring A induces an action G on $\mathrm{Spec}(A)$.[2]

[1] Unfortunately, the literature is not consistent on this convention when G is abelian; e.g. when $G := \mathbb{G}_m$.

[2] We leave it as an exercise to the reader to check this correspondence.

In the case of the standard $\mathbb{G}_m$-action on $\mathbb{A}^1 = \mathrm{Spec}(k[x])$, the weight spaces are given by $k[x]_\lambda = 0$ for $\lambda > 0$ and $k \cdot x^{-\lambda}$ for $\lambda < 0$. Hence, the weight decomposition is

$$k[x] = \bigoplus_{i \le 0} k \cdot x^{-i}.$$

Definition 2.4. Assume an algebraic group G acts on schemes X and Y. A morphism $f : X \to Y$ is *equivariant* if the diagram commutes:

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma_X} & X \\ \Big\downarrow{\scriptstyle \mathrm{id}_G, f} & & \Big\downarrow{\scriptstyle f} \\ G \times Y & \xrightarrow{\sigma_Y} & Y. \end{array}$$

Example 2.5. Assume G acts on schemes X and Y. Then the product $X \times Y$ admits a diagonal G-action, where $g \cdot (x, y) = (gx, gy)$ for all $g \in G(k)$, $x \in X(k)$ and $y \in Y(k)$. The projection morphism

$$X \times Y \to Y$$

is G-equivariant. This simple construction appears frequently in the K-stability literature, when $Y = \mathbb{A}^1$ and $G = \mathbb{G}_m$.

3. Linearized line bundles

A linearization of a line bundle L on a scheme X with a group action is a group action on L extending the group action on X. This is made precise in the following two (equivalent) definitions.

Definition 3.1. Let G be an affine algebraic group acting on a scheme X, L a line bundle on X, by which we mean a rank 1 locally free sheaf, and $\mathbb{L} := V(L)$ the total space of the line bundle.

(1) A *linearization* of $\mathbb{L}$ is bundle isomorphism[3] $\Sigma : G \times \mathbb{L} \to \mathbb{L}$ such that the diagram

$$\begin{array}{ccc} G \times \mathbb{L} & \xrightarrow{\Sigma} & \mathbb{L} \\ \Big\downarrow & & \Big\downarrow \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

commutes and Σ induces an action of G on $\mathbb{L}$.

(2) A *linearization* of L is an isomorphism

$$\sigma^* L \to p_2^* L,$$

[3] The terms *bundle isomorphism* in this setting may seem unusual, since $G \times \mathbb{L}$ is a vector bundle on $G \times X$, while $\mathbb{L}$ is a vector bundle on X. The term means that the induced morphism $G \times \mathbb{L} \to \sigma^* \mathbb{L}$ is an isomorphism of vector bundles on $G \times X$.

where $\sigma : G \times X \to X$ and $p_2 : G \times X \to X$ denote the morphism inducing the action on X and the second projection, such that the cocycle condition

$$(\sigma \times \mathrm{id}_X)^*\Phi = p_{23}^*\Phi \circ (\mathrm{id}_G \times \mu)^*\Phi$$

holds.

A linearization of a vector bundle E on X can be define in an analogous way.

Remark 3.2. As explained in [**MFK94**, Section 1.3], the two definitions in Definition 3.1 are equivalent. To see this, observe that a bundle isomorphism Σ such that the diagram in (1) commutes is equivalent to an isomorphism

$$p_2^*\mathbb{L} := G \times \mathbb{L} \to \sigma^*\mathbb{L}$$

of vector bundles on $G \times X$.[4] Hence, the first part of (2) is the dual of the previous isomorphism. The cocycle condition in (2) is related to the condition in (1) that Σ induces an action of G on $\mathbb{L}$.

Remark 3.3. In Definition 3.1.1, the commutativity of the diagram implies that Σ induces a linear map

$$\mathbb{L}_x \to \mathbb{L}_{g\cdot x}$$

for each $g \in G(k)$ and $x \in X(k)$.

Example 3.4. We consider linearizations of line bundles on $\mathbb{P}^n$.

(1) Consider the action of GL_{n+1} on $\mathbb{P}^n$ induced by the action on $\mathbb{A}^{n+1}$. Viewing the tautological bundle as a sub-bundle

$$\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{A}^{n+1},$$

we see $\mathcal{O}_{\mathbb{P}^n}(-1)$ admits a GL_{n+1}-linearization. Hence, its dual $\mathcal{O}_{\mathbb{P}^n}(1)$ admits a GL_{n+1}-linearization. By taking tensor powers and duals, we conclude $\mathcal{O}(m)$ admits a GL_{n+1}-linearization for any $m \in \mathbb{Z}$.

(2) Consider the action of PGL_{n+1} on $\mathbb{P}^n$. While $\mathcal{O}_{\mathbb{P}^n}(1)$ does not admit a PGL_{n+1}-linearization, its $n+1$ power does! To see the latter, note that the action of PGL_{n+1} induces a linearization of the tangent bundle $T_{\mathbb{P}^n}$ and, hence, of $\mathcal{O}(n+1) \simeq \omega^*_{\mathbb{P}^{n+1}}$.

Example 3.5. Let $X = \mathrm{Spec}(k)$, $L = \mathcal{O}_X = k$, and $\mathbb{L} = \mathbb{A}^1$. A $\mathbb{G}_m$-linearization of L is equivalent to a linear action $\mathbb{G}_m$ on $\mathbb{L} = \mathbb{A}^1$. Such actions are in bijection with group homomorphisms $\mathbb{G}_m \to \mathbb{G}_m$, which are in bijection with $\mathbb{Z}$. Explicitly, $\lambda \in \mathbb{Z}$ induces a linear action $\mathbb{G}_m$ on $\mathbb{L}$ such that

$$a \cdot x = a^\lambda x$$

for all $a \in \mathbb{G}_m(k)$ and $x \in \mathbb{A}^1(k)$.

[4] Here, we are using that $\sigma^*\mathbb{L}$ is the fiber product of $(G \times X) \times_X \mathbb{L}$ via the morphism $G \times X \xrightarrow{\sigma} X$ and $\mathbb{L} \to X$.

EXAMPLE 3.6. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ by $t \cdot (x, y) = (tx, y)$. This induces a $\mathbb{G}_m$-action on $\mathbb{P}^1$, via $t \cdot [x : y] = [tx : y]$ and, hence, the tautological bundle viewed again as a sub-bundle

$$\mathcal{O}_{\mathbb{P}^1}(-1) \subset \mathbb{P}^1 \times \mathbb{A}^2.$$

Since $0 := [0 : 1]$ and $\infty := [1 : 0]$ are fixed points of the action, it follows that the restrictions $\mathcal{O}_{\mathbb{P}^1}(-1)_0$ and $\mathcal{O}_{\mathbb{P}^1}(-1)_\infty$ are $\mathbb{G}_m$-linearized line bundles on a point. We can compute

$$\mathrm{wt}(\mathcal{O}_{\mathbb{P}^1}(-1)_0) = 0 \quad \text{and} \quad \mathrm{wt}(\mathcal{O}_{\mathbb{P}^1}(-1)_\infty) = 1$$

are $\mathbb{G}_m$-linearized line bundles on a point. Note that the difference is -1.

PROPOSITION 3.7. *Let $\mathbb{G}_m$ act on $\mathbb{P}^1$ by $t \cdot [x : y] = [tx : y]$. If $\mathcal{O}_{\mathbb{P}^1}(m)$ admits a $\mathbb{G}_m$-linearization, then*

$$\mathrm{wt}\left(\mathcal{O}_{\mathbb{P}^1}(m)_0\right) - \mathrm{wt}\left(\mathcal{O}_{\mathbb{P}^1}(m)_\infty\right) = m,$$

where $0 := [0 : 1]$ *and* $\infty := [1 : 0]$.

Note that, under the embedding $i : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ defined by $i(x) = [x : 1]$, the above $\mathbb{G}_m$-action on $\mathbb{P}^1$ restricts to the standard $\mathbb{G}_m$-action on $\mathbb{A}^1$ and $i(0) = 0$.

PROOF. The formula holds when $m = 0$, since any linearization is determined by the $\mathbb{G}_m$-action on the nowhere-vanishing section $1 \in \mathcal{O}_{\mathbb{P}^1}$. Hence, for any such linearization, the weights at 0 and ∞ agree.

Next, fix a linearization of $\mathcal{O}_{\mathbb{P}^1}(m)$, where $m \neq 0$. Consider the linearization of $\mathcal{L} := \mathcal{O}_{\mathbb{P}^1}(-1)$ given in the previous example. Using that $\mathcal{M} := \mathcal{O}_{\mathbb{P}^1}(m) \otimes \mathcal{L}^{\otimes m}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}$ and admits an induced $\mathbb{G}_m$-linearization, we compute

$$\begin{aligned}\mathrm{wt}\left(\mathcal{O}_{\mathbb{P}^1}(m)_0\right) - \mathrm{wt}\left(\mathcal{O}_{\mathbb{P}^1}(m)_\infty\right) &= \left(\mathrm{wt}\left(\mathcal{M}_0\right) - \mathrm{wt}\left(\mathcal{M}_\infty\right)\right) - m\left(\mathrm{wt}\left(\mathcal{L}_0\right) - \mathrm{wt}\left(\mathcal{L}_\infty\right)\right) \\ &= 0 - m(-1) = m.\end{aligned}$$

$\square$

APPENDIX B

Intersection numbers and positivity of line bundles

In this appendix, we review the theory of intersection numbers of line bundles and various positivity notions.

Conventions: Throughout, all schemes are defined over an algebraically closed field k of arbitrary characteristic.

1. Intersection numbers

Below, we recall the theory of intersection numbers of line bundles following the approach of Kleiman [**Kle66**]. For detailed proofs, we refer the reader to [**Kle66**] or [**dFEM14**, Section 1.2]. For an alternative topological definition of intersection numbers, see [**Laz04**, Section 1.1.C].

1.1. Definition. Intersection numbers can be defined using the following theorem due to Snapper. For a proof see [**Kle66**, Section 1].

THEOREM 1.1. *If $\mathcal{L}_1, \ldots, \mathcal{L}_r$ are line bundles on a proper scheme X, then the function $\mathbb{Z}^r \to \mathbb{Z}$ defined by*

$$(m_1, \ldots, m_r) \mapsto \chi(X, \mathcal{L}^{m_1} \otimes \cdots \otimes \mathcal{L}^{m_r})$$

is a numeral polynomial of total degree $\leq \dim(X)$.

By a numerical polynomial, mean a polynomial in $\mathbb{Q}[m_1, \ldots, m_r]$ that takes integer values whenever $m_1, \ldots, m_r$ are integers.

DEFINITION 1.2 (Intersection number). Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be line bundles on a proper scheme X, with $\dim(X) \leq r$. The *intersection number*

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_r)$$

is the coefficient of the monomial $m_1 \cdots m_r$ in the polynomial $\chi(X, \mathcal{L}^{m_1} \otimes \cdots \otimes \mathcal{L}^{m_r})$. It follows from the general theory of numerical polynomials that $(\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_r)$ is an [**Kle66**, Proposition 0].

Kleiman considers intersection numbers of the form $(\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_r; \mathcal{F})$, where additionally $\mathcal{F}$ is a coherent sheaf on X [**Kle66**]. The latter agrees with the above definition when $\mathcal{F} = \mathcal{O}_X$. We will not need the extra level of generality in these notes.

1.2. Properties. The properties that intersection numbers satisfy are often more important than their actual definition. The following results are proven in Section 2 of [**Kle66**, Secton 2].

PROPOSITION 1.3. *Let $\mathcal{L}_1, \dots, \mathcal{L}_r$ be line bundles on a proper scheme X, with* $\dim(X) \leq r$.

(1) *If* $\dim(X) < r$, *then* $(\mathcal{L}_1 \cdot \dots \cdot \mathcal{L}_r) = 0$.
(2) *The map* $\mathrm{Pic}(X)^r \to \mathbb{Z}$ *given by*
$$(\mathcal{L}_1, \dots, \mathcal{L}_r) \mapsto (\mathcal{L}_1 \cdot \dots \cdot \mathcal{L}_r)$$
is symmetric and multilinear.
(3) *If $Y_1, \dots, Y_s$ are the r-dimensional irreducible components of X with reduced scheme structure, then*
$$(\mathcal{L}_1 \cdot \dots \cdot \mathcal{L}_r) = \sum_{i=1}^{s} \ell(\mathcal{O}_{X,\eta_i})(\mathcal{L}_1|_{Y_i} \cdot \dots \cdot \mathcal{L}_r|_{Y_i}),$$
where $\eta_i \in X$ is the generic point of Y_i and $\ell(\mathcal{O}_{X,\eta_i})$ denotes the length of the Artinian ring $\mathcal{O}_{X,\eta_i}$.
(4) *(Projection formula) If $f : Y \to X$ is a surjective morphism of proper schemes that is generically finite of degree d, then*
$$(f^*\mathcal{L}_1 \cdot \dots \cdot f^*\mathcal{L}_r) = d(\mathcal{L}_1 \cdot \dots \cdot \mathcal{L}_r).$$
(5) *(Restriction) If $\mathcal{L}_r = \mathcal{O}_X(D)$ for some effective Cartier divisor D on X, then*
$$(\mathcal{L}_1 \cdot \dots \cdot \mathcal{L}_r) = (\mathcal{L}_1|_D \cdot \dots \cdot \mathcal{L}_{r-1}|_D).$$

REMARK 1.4. The theory of intersection numbers on curves and surfaces appears in [**Har77**].

(1) If C is a smooth curve, then the intersection number of a line bundle $\mathcal{L}$ is $\deg(\mathcal{L})$.
(2) If S is a smooth surface and D_1 and D_2 Cartier divisors on S, then
$$(\mathcal{O}_X(D_1) \cdot \mathcal{O}_X(D_2)) = D_1 \cdot D_2,$$
where the latter expression is defined in [**Har77**, V.1].

NOTATION 1.5. In these notes, we will often work with intersection numbers of Cartier divisors.

(1) If $D_1, \dots, D_r$ are Cartier divisors on a proper scheme X with $\dim(X) \leq r$, then we set
$$D_1 \cdot \dots \cdot D_r := (\mathcal{O}_X(D_1) \cdot \dots \cdot \mathcal{O}_X(D_r)).$$
(2) More generally, if we instead assume $D_1, \dots, D_r$ are $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors, then we set
$$D_1 \cdot \dots \cdot D_r := \frac{1}{m^r} (\mathcal{O}_X(mD_1) \cdot \dots \cdot \mathcal{O}_X(mD_r)),$$

where m is a sufficiently divisible positive integer such that $mD_1, \dots, mD_r$ are Cartier divisors. The latter is well defined by Proposition 1.3.2.

EXAMPLE 1.6. Let X be a projective variety of dimension n. If $D_1, \dots, D_n$ are effective Cartier divisors that meat transversely at smooth points, then Proposition 1.3 implies

$$D_1 \cdot \ldots \cdot D_n = \#(D_1 \cap \cdots \cap D_n),$$

where the right hand side denotes the number of points in the intersection.

1.3. Riemann–Roch. A key feature of intersection numbers is that they can be used to describe coefficients of the Hilbert polynomial.

THEOREM 1.7. *If X is a proper scheme of dimension n and $\mathcal{L}$ a line bundle on X, then $\chi(X, \mathcal{L}^m)$ is a polynomial in m of the form*

$$\chi(X, \mathcal{L}^m) = \frac{(\mathcal{L}^n)}{n!} m^n + \text{ lower order terms.}$$

Furthermore, if X is normal, then

$$\chi(X, \mathcal{L}^m) = \frac{(\mathcal{L}^n)}{n!} m^n - \frac{(\mathcal{L}^{n-1} \cdot K_X)}{2(n-1)!} m^{n-1} + \text{ lower order terms.}$$

The first part of the theorem can be deduced easily from 1.1 and the definition of the intersect numbers; see [**dFEM14**, Remark 1.2.9]. The second part requires more work and can be proven using Grothendieck–Riemann–Roch Theorem; see [**BHJ17**, Theorem A.I]

2. Positivity of line bundles

In this section, we recall various notions of positivity for line bundles. For a more in depth treatment of this topic, see [**Laz04**, Chapters 1 and 2].

2.1. Ample, semi-ample, and nef. Recall, a line bundle $\mathcal{L}$ on a proper scheme X is ample if the complete linear system $|\mathcal{L}^{\otimes m}|$ induces an embedding $X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L}^{\otimes m})^*)$ for $m > 0$ sufficiently large. The following are weaker notions of this condition.

DEFINITION 2.1. A line bundle $\mathcal{L}$ on a proper scheme X is

(1) *semi-ample* if $\mathcal{L}^{\otimes m}$ is globally generated for $m \gg 0$.
(2) *nef* if $\mathcal{L} \cdot C \geq 0$ for all irreducible curves $C \subset X$, where

$$\mathcal{L} \cdot C := \deg(\mathcal{L}|_C),$$

REMARK 2.2 (Divisors). The above definitions also make sense for divisors.

(1) A Cartier divisor D on X is *semi-ample* or *nef* if the corresponding property holds for $\mathcal{O}_X(D)$, and we set

$$D \cdot C := \deg(\mathcal{O}_X(D)|_C))$$

(2) More generally, a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on X is *semi-ample* or *nef* if the corresponding properties holds for $\mathcal{O}_X(mD)$ for some $m > 0$ such that mD is a Cartier divisor.

REMARK 2.3. It is straightforward to see that

$$\text{ample} \;\Rightarrow\; \text{semi ample} \Rightarrow\; \text{nef}\ ,$$

where the second implication uses that if $\mathcal{L}$ is a semi-ample line bundle on X and $C \subset X$, then $\mathcal{L}|_C$ is semi-ample and, hence, has positive degree.

If $\mathcal{L}$ is ample, then $\mathcal{L} \cdot C > 0$ for every curve $C \subset X$. While the converse does not hold in general, the statement can be modified slightly to give a criterion for ampleness, where one uses certain limits of curve classes[**Laz04**, Theorem 1.4.29].

DEFINITION 2.4 (Numerical equivalence). Two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X are *numerically equivalent*, denoted $\mathcal{L}_1 \equiv \mathcal{L}_2$, if $\mathcal{L}_1 \cdot C = \mathcal{L}_2 \cdot C$ for every irreducible curve $C \subset X$.

2.2. Bigness. Before defining the notion of a big line bundle, we state the following lemma.

LEMMA 2.5. *If $\mathcal{L}$ is a line bundle on a projective scheme X of dimension n, then there exists a constant $C := C(\mathcal{L})$ such that*

$$\dim H^0(X, \mathcal{L}^{\otimes m}) \leq Cm^n$$

for all positive integers m.

PROOF. Since X is projective, there exist ample Cartier divisors A and B on X such that $\mathcal{L} \simeq \mathcal{O}_X(A - B)$. The injection $\mathcal{O}_X(m(A - B)) \hookrightarrow \mathcal{O}_X(mA)$ of sheaves induces an injection

$$H^0(X, \mathcal{L}^{\otimes m}) \hookrightarrow H^0(X, \mathcal{O}_X(mA)).$$

Since $H^0(X, \mathcal{O}_X(mA))$ agrees with a polynomial of degree n when $m \gg 0$, the result follows. □

DEFINITION 2.6. The *volume* of a line bundle $\mathcal{L}$ on a projective variety X of dimension n is the limsup

$$\mathrm{vol}(\mathcal{L}) := \limsup_{m\to\infty} \frac{h^0(X, \mathcal{L}^{\otimes m})}{m^n/n!}.$$

We say $\mathcal{L}$ is *big* if $\mathrm{vol}(L) > 0$.

REMARK 2.7. We state a number of properties of the volume.

(1) If $\mathcal{L}$ is nef, then $\mathrm{vol}(\mathcal{L}) = \mathcal{L}^n$ by [**Laz04**, Corollary 1.4.41]. Additionally, if $\mathcal{L}$ is ample, then $\mathrm{vol}(\mathcal{L}) > 0$.
(2) The limsup in the definition of volume is in fact a limit. This is not immediately obvious; see e.g. [**Laz04**, Section 11.4].

(3) If a is a positive integer, then

$$\mathrm{vol}(\mathcal{L}^{\otimes a}) = a^n \mathrm{vol}(\mathcal{L}).$$

This follows from the definition and that the volume is a limit.

REMARK 2.8 (Divisors). As before, the definitions naturally extends to divisors.

(1) A Cartier divisor D is *big* $\mathrm{vol}(D) := \mathrm{vol}(\mathcal{O}_X(D)) > 0$.
(2) A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor is *big* if $\mathrm{vol}(D) := \frac{1}{m^n}\mathrm{vol}(\mathcal{O}_X(mD))$, where m is a positive integer such that mD is a Cartier divisor.

The following lemma shows that bigness of D can be characterized by the map $|mD|$ for $m > 0$ sufficiently large [**KM98**, Lemma 2.60]

PROPOSITION 2.9. *Let D be a Cartier divisor on projective variety X. The following are equivalent:*

(1) D is big,
(2) there exists an ample divisor A, effective divisor E, and $m > 0$ such that $mD \sim A + E$,
(3) The rational map $\phi_{mD} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(mD))^*)$ is birational onto its image for $m \gg 0$.

PROOF. To see the implication (1) $\Rightarrow$ (2), fix an effective ample divisor A on X and consider the exact sequence

$$0 \to H^0(X, \mathcal{O}_X(mD - A)) \to H^0(X, \mathcal{O}_X(mD)) \to H^0(A, \mathcal{O}_X(mD)|_A).$$

Since D is big and $H^0(A, \mathcal{O}_X(mD)|_A) = O(m^{n-1})$, it follows that $H^0(X, \mathcal{O}_X(mD - A)) \neq 0$ for $m \gg 0$. Thus, there exists an effective divisor $E \in |mD - A|$ when $m \gg 0$ and, hence, (2) holds.

The implication (2) $\Rightarrow$ (3) is straightforward. To verify (3) $\Rightarrow$ (1), let Y denote the closure of the image of ϕ_{mD}. Since $h^0(Y, \mathcal{O}_Y(k))$ agrees with a polynomial of degree n for $k \gg 0$ and ϕ_{mD}^* induces an injection

$$H^0(Y, \mathcal{O}_X(k)) \hookrightarrow H^0(X, \mathcal{O}_X(mkD)),$$

so (1) holds. $\square$

EXAMPLE 2.10. Let us compute the volume of the line bundles on $X := B_p\mathbb{P}^2$. First, note that $\mathrm{Pic}(X) \simeq \mathbb{Z}^2$ and is freely generated by $\mathcal{O}_X(H)$ and $\mathcal{O}_X(E)$, where H is the birational transform of a line not passing through p and E is the exceptional divisor. Note that $H \cdot H = 1$, $E \cdot H = 0$, and $E^2 = 0$.

(1) $xH - yE$ is nef if and only if $x \geq 0$ and $y \geq 0$.

To see the forward implication, note that

$$(xH - yE) \cdot H = x.$$

Additionally, if we fix a line $p \in L \subset \mathbb{P}^2$, then $H \sim \pi^* L = \tilde{L} + E$ and, hence,

$$(xH - yE) \cdot \tilde{L} = (xH - yE)(H - E) = x - y.$$

Thus, if $xH - yE$, then $x \geq 0$ and $x - y \geq 0$.

For the reverse implication it suffices to show both H and $H - E$ are nef, since then any linear combination of the two is nef. The statement is true, since $|H|$ and $|H - E|$ are base point free.

(2) $\operatorname{vol}(xH - yE) = \begin{cases} x^2 - y^2 & \text{if } x \geq 0 \text{ and } x - y \geq 0 \\ x^2 & \text{if } x \geq 0 \text{ and } y \leq 0 \end{cases}$

If $x \geq 0$ and $x - y \geq 0$, then $xH - yE$ is nef, and, hence,

$$\operatorname{vol}(xH - yE) = (xH - yE)^2 = x^2 - y^2.$$

For the next region, note that if $a, b \in \mathbb{N}$, then

$$H^0(X, \mathcal{O}_X(aH + bE)) = H^0(X, \mathcal{O}_X(aH)).$$

One way to see this is to note that if $D \in |aH + bE|$, then $D \cdot E = (aH + bE) \cdot E = -b$, so $bE \leq D$. Hence, when $x \geq 0$ and $y \leq 0$,

$$\operatorname{vol}(xH - yE) = \operatorname{vol}(xH) = x^2.$$

APPENDIX C

Singularities

1. Singularities of the Minimal Model Program

In this section, we recall definitions and properties of singularities appearing in the Minimal Model Program. See [**KM98**, Section 2.3] for a brief summary of the relevant definitions. For a more detailed reference, see [**dFEM14**, Chapter 3].

Conventions: Throughout, all schemes are defined over an algebraically closed field k of characteristic 0.

1.1. Divisors.

We briefly recall some terminology for divisors.

Let X be a normal variety. A *divisor* D on X is a formal sum $D = \sum_{i=1}^r a_i D_i$, where each D_i is a prime divisor on X and $a_i \in \mathbb{Z}$. A $\mathbb{Q}$-*divisor* D on X is a formal sum $D = \sum_{i=1}^r a_i D_i$, where $a_i \in \mathbb{Q}$. It is *effective* if $a_i \geq 0$ for all i. The *support* of $D = \sum a_i D_i$ is the closed subscheme

$$\mathrm{Supp}(D) := \bigcup_{a_i \neq 0} D_i \subset X.$$

Two $\mathbb{Q}$-divisors D and D' are $\mathbb{Q}$-*linearly equivalent* if there exists an integer $m > 0$ such that mD and mD' are divisors and $mD \sim mD'$, where $\sim$ denotes linear equivalence. A $\mathbb{Q}$-divisor D is $\mathbb{Q}$-*Cartier* if there exists exists an integer $m > 0$ such that mD is a Cartier divisor.

Let $f : Y \to X$ be a proper birational morphism and D be a $\mathbb{Q}$-divisor on X. If D is $\mathbb{Q}$-Cartier, we set

$$f^*(D) := \frac{1}{m} f^*(mD),$$

where m is a positive integer such that mD is Cartier divisor. Note that, even if D has integral coefficients, f^*D may have fractional coefficients.

A divisor D on X is *simple normal crossing*, which is abbreviated *snc*, if X is smooth and at each $p \in \mathrm{Supp}(D)$ and there exists local coordinates[1] $x_1, \ldots, x_n$ at p such that D is locally defined at p by $x_1 \cdot \ldots \cdot x_r$ for some $1 \leq r \leq n$.

A *log resolution* of (X, D), where X is a normal variety and D a $\mathbb{Q}$-divisor on X is a proper birational morphism $Y \to X$ such that

- Y is smooth,
- $\mathrm{Exc}(f)$ is pure codimension 1, and

[1]By local coordinates at p, we mean $x_1, \ldots, x_n \in \mathcal{O}_{X,p}$ such that their images $\overline{x}_1, \ldots, \overline{x}_n \in \mathfrak{m}_/\mathfrak{m}_p^2$ form a basis for $\mathfrak{m}_p/\mathfrak{m}_p^2$.

- $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}(D))$ is snc.

As a consequence of results of Hironoka [**Hir64**], log resolutions always exist in characteristic 0.

1.2. Canonical divisor. The singularities notions in the Minimal Model Program are defined using the canoncial divisor on a variety and how it transforms under birational morphisms.

Definition 1.1 (Canonical divisor). Let X be a normal variety. A *canonical* divisor on a normal variety X is a divisor K_X on X such that

$$\omega_U \simeq \omega_U(K_X|_U),$$

where U is the smooth locus of X. Since $\mathrm{codim}_X(X \setminus U) \geq 2$ by the normallity assumption on X, any two canonical divisors of X are linearly equivalent.

Example 1.2. We give two examples of the canonical divisor.

(1) If $H \subset \mathbb{P}^n$ is a hyperplane, then $K_{\mathbb{P}^n} = -(n+1)H$.
(2) $X \subset \mathbb{P}^n$ is a normal hypsurface of degree d, then

$$K_X = -(n+1-d)H|_X,$$

where $H \subset \mathbb{P}^n$ is a hyperplane. (When $d = 1$, we assume $H \neq X$ so that the restriction $H|_X$ is a well-defined divisor).

Lemma 1.3. *If $f : Y \to X$ is a proper birational morphism of normal varieties and K_Y is a canonical divisor on Y, then $f_* K_Y$ is a canonical divisor on X.*

Proof. Let $\mathrm{Exc}(f) \subset Y$ denote the exceptional locus of f. Since the morphism

$$V := Y \setminus \mathrm{Exc}(f) \to U := X \setminus f(\mathrm{Exc}(f)).$$

is an isomorphism $(f_* K_Y)|U$ is a canonical divisor on U. Since $\mathrm{codim}_X(U) \geq 2$ by [**Har77**, Proof of Lemma V.5.1], it follows that $f_* K_Y$ is a canonical divisor on X. $\square$

By the previous lemma, if $f : Y \to X$ is a proper birational morphism with X and Y normal, then we can choose canonical divisors K_Y and K_X such that

$$K_Y = f_* K_X.$$

Whenever we are in such a setup, we will always assume this choice has been made.

Definition 1.4 (Relative canonical divisor). Let $f : Y \to X$ be a proper birational morphism of normal varieties with K_X $\mathbb{Q}$-Cartier. The *relative canonical divisor* of f is the $\mathbb{Q}$-divisor

$$K_{Y/X} := K_Y - f^* K_X.$$

Since in the above formula K_Y and K_X are chosen so that $f_* K_Y = K_X$, it follows that $K_{Y/X}$ is exceptional, by which we mean $f_* K_{Y/X} = 0$.

Example 1.5. We list a few examples of the relative canonical divisor.

(1) Let X be a smooth variety, $Z \subset X$ a smooth subvariety of codimension r, and
$$f : Y := B_Z X \to X$$
denote the blowup of X along Z with exceptional divisor E. Then
$$K_{Y/X} = (r-1)E.$$

This formula can be deduced from the following local computation. To simplify the computation, let us assume Z is a point $x \in X$ and, hence, $r = n := \dim(X)$. In this case, choose local coordinates
$$x_1, \ldots, x_n \in \mathcal{O}_{X,x},$$
and a point $y \in E$ such that there are coordinates
$$y_1, \ldots, y_n \in \mathcal{O}_{Y,y},$$
where $f^*x_1 = y_1$ and $f^*x_i = y_1 y_i$ for $2 \leq i \leq n$. Then we see
$$f^*dx_1 = dy_1 \quad \text{and} \quad f^*dx_i = y_1 dy_i + y_i dy_1$$
and, hence,
$$f^*dx_1 \wedge \ldots \wedge dx_n = y_1^{n-1} dy_1 \wedge \ldots \wedge dy_n.$$
Therefore, $K_{Y/X} = (n-1)E$ in a neighborhood of y. Since we known $K_{Y/X}$ is exceptional and E is the sole exceptional divisor of f, we conclude the formula holds.

(2) If $f : Y \to X$ be a proper birational morphism of smooth varieties, then $K_{Y/X}$ is effective. Indeed, by [**Har77**, Proposition 8.11], there is a natural exact exact sequence
$$f^*\Omega_X \to \Omega_Y \to \Omega_{Y/X} \to 0.$$
Since $f^*\Omega_X \to \Omega_Y$ is an isomorphism over $V := Y \setminus \operatorname{Exc}(f)$, taking the top exterior powers gives an injective map $f^*\omega_X \hookrightarrow \omega_Y$. Hence, $K_{Y/X}$ is effective.

In fact, $\operatorname{Supp}(K_{Y/X}) = \operatorname{Exc}(f)$. This is shown by reducing the problem to understanding a sequence of blowups along smooth centers as in the first example. See [**KM98**, Corollary 2.31] for details.

(3) Let $h \in k[x_0, \ldots, x_n]$ be a homogenous polynomial of degree d with $n \geq 3$. Assume $H := \{h = 0\} \subset \mathbb{A}^{n+1}$ has an isolated singularity at 0, which implies H is the cone over a smooth degree d hypersurface in $\mathbb{P}^n$.

Consider the commutative diagram
$$\begin{tikzcd} \widetilde{H} \arrow[r, hook] \arrow[d, "g"] & Y \arrow[d, "f"] \\ H \arrow[r, hook] & \mathbb{A}^{n+1} \end{tikzcd},$$

where f is the blowup of $\mathbb{A}^{n+1}$ at 0 and $\widetilde{H}$ is the strict transform on H on Y. Let $E \subset Y$ denote the exceptional divisor and note that $\widetilde{H} \cap E$ is isomorphic to the smooth hypsurface in $\mathbb{P}^n$ cut out by h

To compute the relative canonical divisor of $\widetilde{H} \to H$, we use adjunction. The latter says that if $D \subset X$ is a smooth divisor on a smooth variety, then the natural map $\omega_X(D)|_D \to \omega_D$ is an isomorphism; see [**Har77**, Proposition 8.20] for a similar statement. Hence,

$$(K_X + D)|_D = K_D.^{2}$$

Returning to our computation, adjunction gives

$$(K_Y + \widetilde{H})|_{\widetilde{H}} = K_{\widetilde{H}} \quad \text{and} \quad (K_{\mathbb{A}^{n+1}} + H)|_H = K_H.$$

Thus,

$$\begin{aligned} K_{\widetilde{H}/H} &= (K_Y + \widetilde{H})|_{\widetilde{H}} - g^*((K_{\mathbb{A}^{n+1}} + H)|_H) \\ &= (K_Y + \widetilde{H} - f^*(K_{\mathbb{A}^{n+1}} + H))|_{\widetilde{H}} \\ &= (K_{Y/\mathbb{A}^{n+1}} + \widetilde{H} - f^*H)|_{\widetilde{H}} \\ &= (n-d)E|_{\widetilde{H}}. \end{aligned}$$

Thus, when $d > n$, $K_{\widetilde{H}/H}$ fails to be effective. As we will see, this is related to the fact that the singularity of H gets worse as d gets large.

1.3. Singularities of varieties. The relative canonical divisor gives rise to a measure of singularities of a variety.

Definition 1.6. A variety X is called klt (resp., lc) if

(1) X is normal,
(2) K_X is $\mathbb{Q}$-Cartier,
(3) and $K_{Y/X}$ has coefficients > -1 (resp., ≥ 1) for all proper birational morphisms $f : Y \to X$ with Y normal.

By Lemma 1.15, (3) is equivalent to the condition that $K_{Y/X}$ has coefficients > -1 (resp., ≥ -1) for a single log resolution $f : Y \to X$ of X.

Remark 1.7 (Terminology). *Klt* and *lc* are abbreviations for Kawamata log termina and log canonical, respectively. These are classes of singularities that appear naturally in the Minimal Model Program.

Example 1.8. The following examples show that klt and lc are natural notions of singularities.

(1) A smooth variety X is klt. This follows from Example 1.5.1 or taking the log resolution to be the identity morphism.

[2] For the restriction $(K_X + D)|_D$ to be a well defined divisor on D, we choose K_X so that $D \not\subset \mathrm{Supp}(K_X + D)$.

(2) Let $H \subset \mathbb{A}^{n+1}$ be the hypersurface in Example 1.5.3. Since $\widetilde{H} \to H$ is a log resolution of H and

$$K_{\widetilde{H}/H} = (n-d)E|_{\widetilde{H}},$$

H is klt (resp., lc) if and only if $d \leq n$ (resp., $d \leq n+1$). Note that $d \leq n$ precisely when H is the cone over a smooth Fano variety.

(3) Let X be a smooth projective variety and L an ample Cartier divisor on X. The affine cone over X with respect to L is

$$C_a(X, L) := \operatorname{Spec}\big(\bigoplus_{m \in \mathbb{N}} H^0(X, mL).$$

By [**Kol13a**, Section 3.1], $K_{C_a(X,L)}$ is $\mathbb{Q}$-Cartier if $K_X \sim_{\mathbb{Q}} rL$ for some $r \in \mathbb{Q}$. Additionally, $C_a(X, L)$ is klt (resp., lc) if and only if $r < 0$ (resp., < 0).

Thus, the cone over a Fano variety with respect to the anti-canonical divisor is klt. The cone over a Calabi-Yau variety with respect to any polarization is lc, but not klt.

(4) Let $G \subset \mathrm{GL}_n$ be a finite group. By [**Kol13a**, pg. 103], the quotient of $\mathbb{A}^n$ by G, which is defined as

$$X := \mathbb{A}^n/G := \operatorname{Spec}(k[x_1, \dots, x_n]^G),$$

where $k[x_1, \dots, x_n]^G := \{f \in k[x_1, \dots, x_n] \mid g \cdot f = f \text{ for all } g \in G\}$ is the ring of invariants.

For a simple example, let $G \subset \mathrm{GL}_2$ denote the subgroup generated by

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{m-1} \end{pmatrix},$$

where ξ is a primitive m-th root of unity. Then

$$\mathbb{A}^2/G := \operatorname{Spec} k[x, y]^G = \operatorname{Spec}(k[x^m, y^m, xy]),$$

Note that

$$k[x^m, y^m, xy] \simeq k[a, b, c]/(ab - c^m).$$

(5) If X is a normal toric variety such that K_X is $\mathbb{Q}$-Cartier, then X is klt.

Remark 1.9 (Properties of lc and klt). The following diagram shows the relation between, klt, lc, rational, and CM singularities.

$$\begin{array}{ccccc} \text{klt} & \Longrightarrow & \text{rational} & \Longrightarrow & \text{CM} \\ \Downarrow & & & & \\ \text{lc} & & & & \end{array}.$$

1.4. Singularities of pairs. The notions of singularities in the MMP are frequently used in the "log" setting in which one considers the data of a variety with an effective divisor.

DEFINITION 1.10 (Pairs). A *pair* (X, Δ) is a normal variety X and an effective $\mathbb{Q}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

DEFINITION 1.11 (Crepant pullback). Let (X, Δ) be a pair and $f : Y \to X$ be a proper birational morphism with Y normal. The *crepant pullback* of Δ is the $\mathbb{Q}$-divisor Δ_Y on Y such that

$$K_Y + \Delta_Y = f^*(K_X + \Delta).$$

Note that if $f_*\Delta_Y = \Delta$.

While K_Y and K_X are only defined up to linear equivalence, Δ_Y is in fact uniquely determined. Indeed, since we alway assume $f_*K_Y = K_X$, if we were to replace K_Y with $K_Y + \mathrm{div}_Y(\phi)$ for some $\phi \in K(Y)$, then we would need to replace K_X with $K_X + \mathrm{div}_X(\phi)$ and Δ_Y would be unchanged.

EXAMPLE 1.12. We give two examples of the crepant pullback of a pair.

(1) If $\Delta = 0$, then $\Delta_Y = -K_{Y/X}$.
(2) Let $H \subset \mathbb{A}^{n+1}$ be the degree d hypersurface in Example 1.5.3 and set $\Delta = cH$ for some $c \in \mathbb{Q}_{\geq 0}$. Note that f is a log resolution of $(\mathbb{A}^{n+1}, \Delta)$ and

$$\Delta_Y = -K_{Y/\mathbb{A}^{n+1}} + f^*D = -nE + cd\widetilde{H}.$$

Hence, Δ_Y has coefficients < 1 (resp., ≤ 1) when $c \leq \frac{n+1}{d}$.

The divisor Δ_Y can be viewed as a modification of the pullback of Δ that takes into account the relative canonical divisor of f. Its coefficients will be used to define the following singularity notions.

DEFINITION 1.13. A pair (X, Δ) is *klt* (resp., *lc*) if the crepant pullback Δ_Y has coefficients < 1 (resp., ≤ 1) for all proper birational morphisms $f : Y \to X$ with Y normal.

EXAMPLE 1.14. We list a number of examples, which show that klt and lc are reasonable measures of singularities for pairs.

(1) A pair $(X, 0)$ is klt (resp., lc) if and only if X is klt (resp., lc). This follows immediately from the definition, since $\Delta_Y = -K_{Y/X}$ when $\Delta = 0$.
(2) Consider a pair

$$\big(X, \Delta := \sum b_i \Delta_i\big),$$

where X is smooth and $\mathrm{Supp}(\Delta)$ is snc. The pair is klt (resp., lc) if and only if $b_i < 1$ (resp., ≤ 1) for all i. This follows from a local computation [**KM98**, Corollary 2.31], which is similar to Example 1.5.1.
(3) A frequent first example is that $(\mathbb{A}^2, c\{x^2 - y^3 = 0\})$ is lc if and only if $0 \leq c \leq 5/6$. We leave the computation as an exercise for the reader.

(4) If $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with $f(0) = 0$, then $(\mathbb{A}^n_{\mathbb{C}}, c\{f = 0\})$, is klt in a neighborhood of $0 \in \mathbb{A}^n_{\mathbb{C}}$ if and only if

$$\frac{1}{|f|^{2c}}$$

is locally integrable in a neighborhood of 0.

The above relation intuitively makes sense, since if f vanishes to a higher degree at 0, the singularities of $\{f = 0\}$ should be "bad" and $1/|f(x)| \to \infty$ very quickly as $x \to 0$. To prove the relation, one computes the integral

$$\int_{B_0(\epsilon)} \frac{1}{|f|^{2c}}$$

by taking a log resolution $Y \to X$ and equating the previous integral to an integral on Y. The Jacobian term in the integral on Y will involve the coefficients of $K_{Y/X}$. See for example [**Mus11**, pg. 7] for a detailed proof.

Definition 1.13 as stated might seem difficult to check, since it requires taking all birational morphism $Y \to X$ with Y normal. Fortunately, it suffices to consider a single log resolution.

Lemma 1.15. *A pair (X, Δ) is klt (resp., lc) if and only if there exists a log resolution $f : Y \to X$ such that Δ_Y has coefficients < 1 (resp., ≤ 1).*

Proof sketch. See [**KM98**, Corollary 2.31] for a complete proof. Below we sketch the main idea.

Assume $f : Y \to X$ is a log resolution such that Δ_Y has coefficients < 1. Let $f' : Y' \to X$ be a proper birational morphism with Y' normal. We aim to show $\Delta_{Y'}$ has coefficients < 1.

The trick is that we can always find a log resolution $Z \to X$ such that the diagram

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "g'"] & \\ Y \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

commutes. Now, note that Δ_Z, which denotes the crepant pullback of Δ under $f \circ g$ is the crepant pullback of Δ_Y via g, since

$$K_Z + \Delta_Z = g^* f^*(K_X + \Delta) = g^*(K_Y + \Delta_Y).$$

Since Δ_Y has coefficients < 1, Example 1.14.1 implies Δ_Z has coefficients < 1. Since Δ_Z is also the crepant pullback of $\Delta_{Y'}$ via g' (by a similar argument as above), $g'_* \Delta_Z = \Delta_{Y'}$. Hence, $\Delta_{Y'}$ has coefficients < 1. □

1.5. Log canonical thresholds. Using the notion of lc pairs, we discuss the follow measure of singularities for a divisor.

DEFINITION 1.16 (Log canonical threshold). Let X be a klt variety. The *log canonical threshold* (lct) of an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on X is

$$\operatorname{lct}(X, D) := \sup\{c \in \mathbb{Q}_{\geq 0} \mid (X, cD) \text{ is lc}\}.$$

Note that $\operatorname{lct}(X, D)$ is always positive. To see this, take a log resolution $\mu : Y \to X$ of (X, D). Since X is klt, $K_Y - \mu^* K_X$ has coefficients > -1. Hence, for $0 < c \ll 1$, $K_Y - \mu^*(K_X - cD)$ has coefficients ≥ -1, and (X, cD) is lc.

EXAMPLE 1.17. Smaller values for $\operatorname{lct}(X, D)$ correspond to worse singularities. This can be seen in the following examples whose computation follows immediately from Example 1.14.

(1) If X be a smooth variety and $D := \sum_{i=1}^{r} b_i D_i$ be an effective $\mathbb{Q}$-divisor with $\operatorname{Supp}(D_1 + \cdots + D_r)$ snc, then
$$\operatorname{lct}(X, D) = \min_{i=1,\ldots,r} \frac{1}{b_i}.$$
(2) The lct of the cusp is
$$\operatorname{lct}(\mathbb{A}^2, \{x^2 - y^3 = 0\}) = \frac{5}{6},$$
while by the previous examples
$$\operatorname{lct}(\mathbb{A}^2, \{x^2 - y^2 = 0\}) = 1.$$
This corresponds to the intuition that the cusp $\{^2 - y^3 = 0\}$ is more singular than two intersecting lines $\{x^2 - y^2 = (x + y)(x - y) = 0\}$.

(3) If $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with $f(0) = 0$, then
$$\operatorname{lct}_0(\mathbb{A}^n_{\mathbb{C}}, \{f = 0\}) = \sup\left\{c \in \mathbb{Q}_{>0} \mid \frac{1}{|f|^{2c}} \text{ is locally integrable at } 0\right\},$$
where lct_0 denotes the lct in an sufficiently small open neighborhood of 0.

PROPOSITION 1.18. *If X is a klt variety and D an effetive $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, then*

$$\operatorname{lct}(X, D) = \min_{E} \frac{A_X(E)}{\operatorname{ord}_E(D)},$$

where the minimum runs through all divisors E over X.

Furthermore, it suffices to take the minimum over all prime divisors $E \subset \operatorname{Supp}(\mu^* D)$ on a single log resolution $\mu : Y \to X$ of (X, D).

Recall, a divisor E over X is the data of a proper birational morphism $Y \to X$ with Y a normal variety and a prime divisor $E \subset Y$. Additionally,

$$A_X(E) := \operatorname{coeff}_E(K_{Y/X}) \quad \text{and} \quad \operatorname{ord}_E(D) := \operatorname{coeff}_E(\mu^* D).$$

Eqivalently, $\mathrm{ord}_E(D) = \frac{1}{m}\mathrm{ord}_E(f)$, where m is a positive integer such that mD is Cartier and $f \in \mathcal{O}_{X,\mu(E}$ a local defining equation of mD at the generic point of $\mu(E)$. Note that when $E \notin \mathrm{Supp}(\mu^*D)$, then $\mathrm{ord}_E(D) = 0$ and $\frac{A_X(E)}{\mathrm{ord}_E(D)} = +\infty$ by convention.

PROOF. Fix a proper birational morphism $\mu : Y \to X$ with Y a normal variety. Observe that if $E \subset Y$ is a prime divisor, then

$$\mathrm{coeff}_E(K_Y - \mu^*(K_X + cD)) = \mathrm{coeff}(K_{Y/X}) - c\,\mathrm{coeff}\mu^*D = A_X(E) + 1 - c\,\mathrm{ord}_E(D).$$

Thus, the following are equivalent:

(1) $K_Y - \mu^*(cD)$ has coefficients ≥ -1
(2) $c \leq \frac{A_X(E)}{\mathrm{ord}_E(D)}$ for all prime divisors $E \subset \mathrm{Supp}(\mu^*D)$.

Since (X, cD) is lc if and only if $K_Y - \mu^*(cD)$ has coefficients ≥ -1 for all proper birational morphism $Y \to X$ with Y normal, we see

$$\mathrm{lct}(X, D) = \inf_E \frac{A_X(E)}{\mathrm{ord}_E(D)},$$

where the infimum runs through all prime divisors E over X.

Now, assume $Y \to X$ is a log resolution of (X, D). Since (X, cD) is lc if and only if $K_Y - \mu^*(cD)$ has coefficients ≥ -1 for this specific Y by Lemma 1.15,

$$\mathrm{lct}(X, D) = \inf_{E \subset Y} \frac{A_X(E)}{\mathrm{ord}_E(D)}$$

and it is a minimum since we only need to take the prime divisors $E \subset \mathrm{Supp}(\mu^*D)$ and the latter is a finite collection. $\square$

REMARK 1.19. In the setting of Proposition 1.18, it is also the case that

$$\mathrm{lct}(X, D) = \inf_v \frac{A_X(v)}{v(D)},$$

where the infimum runs through all valuations $v \in \mathrm{Val}_X$ with $A_X(v) < \infty$ [**JM12**, **Bou15**]. Indeed, this follows from [**JM12**, Lemma 6.7] when X is smooth and can be deduced from [**Bou15**, Theorem 1.1] in general.

REMARK 1.20. We list a few simple properties of the log canonical threshold.

(1) It follows immediately from the definition that for any $a \in \mathbb{Q}_{>0}$,

$$\mathrm{lct}(X, aD) = a^{-1}\mathrm{lct}(X, D).$$

(2) If $p \in X$ is a smooth point, then

$$\frac{1}{\mathrm{ord}_p(D)} \leq \mathrm{lct}_p(X, D) \leq \frac{n}{\mathrm{ord}_p(D)},$$

where n is the dimension of X and

$$\mathrm{lct}_p(X, D) := \sup\{c \in \mathbb{Q}_{>0} \mid (X, cD) \text{ is lc in an open neighborhood of } p\}.$$

The first inequality follows from a degeneration argument [**Kol97**, Lemma 8.10]. For the second inequality, let E denote the exceptional divisor of the blowup $B_pX \to X$ and note that

$$\mathrm{lct}_p(X, D) \leq \frac{A_X(E)}{\mathrm{ord}_E(D)} = \frac{n}{\mathrm{ord}_p(D)},$$

where the inequality is by Proposition **??**.

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